Deformations of Toric Singularities

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Introduction

In deformation theory, the following question naturally arises: Given a scheme X with only *mild* singularities, what can we say about the singularities of local deformations of X? For X containing at worst isolated toric singularities, a conjecture by Riemenschneider formulates a possible answer. If X is a scheme over a field with characteristic 0, the conjecture is known to be true. However, for schemes over fields of positive characteristic, the conjecture is not yet known.

Riemenschneiders conjecture

In 1974, Riemenschneider in [Rie74] formulated the following conjecture:

Local deformations of isolated toric singularities contain at worst toric singularities.

Deformations

Let X be of finite type over k and S the spectrum of a local noetherian k-algebra. A *local deformation* of X over S is a pull-back diagram of the form

$$\begin{array}{ccc} X \cong \operatorname{Spec} k \times_S \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} k & \longrightarrow & S, \end{array}$$

where π is a flat morphism. If the isomorphism $X \cong \operatorname{Spec} k \times_S \mathcal{X}$ is clear, we will refer to $\mathcal{X} \xrightarrow{\pi} S$ as a local deformation.

A local deformation $\mathcal{X} \xrightarrow{\pi} S$ is called *versal*, if for every other local deformation $\mathcal{Y} \xrightarrow{\pi'} R$ can be obtained from $\mathcal{X} \xrightarrow{\pi} S$ by basechange,

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow \pi' & & \downarrow \pi \\ R & \longrightarrow & S. \end{array}$$

Toric Singularities

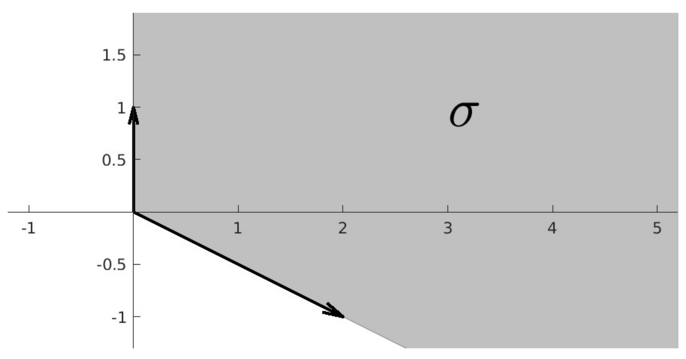
Starting with a strongly convex polyhedral cone, an affine toric variety can be constructed in the following way:

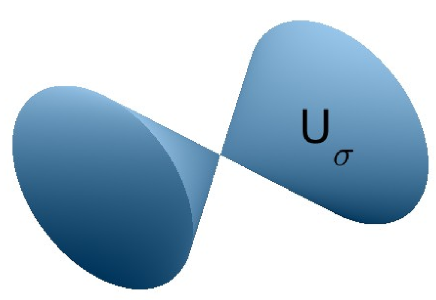
cone $\sigma \to$ dual cone $\sigma^\vee \to$ semigroup $S_\sigma \to$ group algebra $k[S_\sigma] \to$ affine toric variety U_σ

A toric singularity is a singularity occuring in toric varieties.

Example.
The A_n singularities are toric singularities, coming from the two-dimensional cones

$$\sigma = \text{cone}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} n \\ -n+1 \end{pmatrix} \right\} \to U_\sigma = \operatorname{Spec} k[X,Y,Z]/(XZ - Y^n).$$





The cone and associated affine toric variety with singularity A_2.

In order to talk about the isolated singularity as it's own algebro-geometric object, we identify it with the spectrum of the completion of it's local ring.

Example. *Consider the affine toric variety*

$$\sigma = \text{cone}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 11 \\ -8 \end{pmatrix} \right\}$$

$$\to U_\sigma = \operatorname{Spec} k[X_1,X_2,X_3,X_4]/(X_1X_3 - X_2^4, X_2X_4 - X_3^3, X_1X_4 - X_2^3X_3^2).$$

This variety has an isolated singularity at $(0,0,0,0)$, namely the singularity

$$X(4,3) = \operatorname{Spec} k[[X_1,X_2,X_3,X_4]]/(X_1X_3 - X_2^4, X_2X_4 - X_3^3, X_1X_4 - X_2^3X_3^2).$$

Every toric surface can be written as a quotient of a polynomial ring $k[X_1,\ldots,X_e]$ by some ideal I containing among others some polynomials of the form $X_1X_3 - X_2^{a_2}, \ldots, X_{e-2}X_e - X_{e-1}^{a_{e-1}}$. The numbers $a_2,\ldots,a_{e-1}$ suffice to specify the toric surface in question and the singularity of the corresponding toric surface is denoted $X(a_2,\ldots,a_{e-1})$.

Every cyclic quotient singularity is a toric singularity and vice-versa. Therefore, all papers about cyclic quotient singularities can also be read as papers about toric singularities.

History of relevant results

The following table gives an overview of the cases for which Riemenschneiders conjecture has been proven. The last three columns show the dimensions d, the embedding dimension e, and the characteristic of the underlying ground field $\operatorname{char} k$ for which a proof has been found.

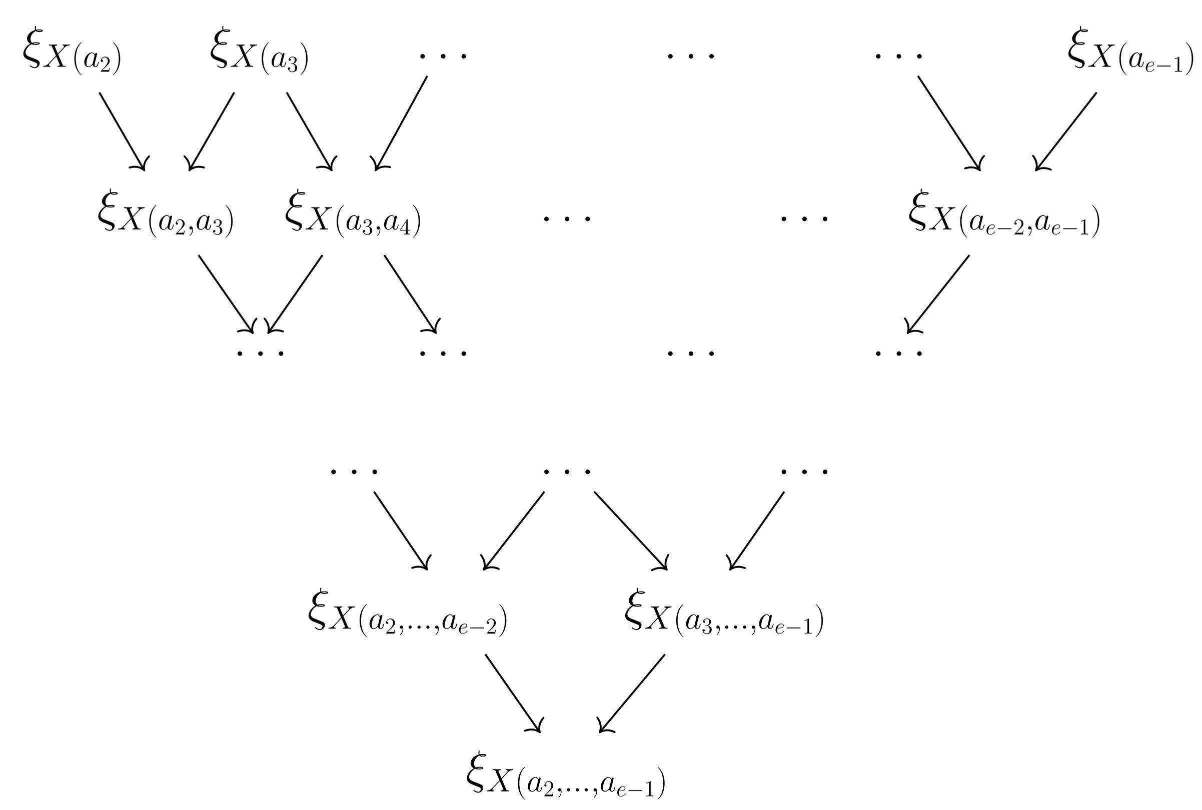
Authors		d	e	$\operatorname{char} k$
Schlessinger	[Sch71]	≥ 3	any	0
Riemenschneider	[Rie74]	2	≤ 5	0
Kollár and Shepherd-Barron	[KSB88]	2	any	0
Liedtke, Martin, and Matsumoto	[LMM]	≥ 3	any	> 0

This means that dim 2, char > 0 is still open!

Additionally, in a series of PhD-theses by Arndt, Brohme, and Hamm, an algorithmic construction of versal deformations of toric surface singularities over $\mathbb{C}$ was found. The versal deformation lets us determine exactly which singularities can be obtained by deforming isolated toric surface singularities.

Finding versal deformations

The construction of a versal deformation $\xi_{X(a_2,\ldots,a_{e-1})}$ of an isolated toric surface singularity $X(a_2,\ldots,a_{e-1})$ over $\mathbb{C}$ that was first given by Arndt in [Arn88] is an iterative procedure. Given a singularity $X(a_2,\ldots,a_{e-1})$, we first construct the versal deformations $\xi_{X(a_2)},\ldots,\xi_{X(a_{e-1})}$. In the next step, the versal deformations $\xi_{X(a_i,a_{i+1})}$ are constructed by combining $\xi_{X(a_i)}$ and $\xi_{X(a_{i+1})}$. Then, $\xi_{X(a_i,a_{i+1})}$ and $\xi_{X(a_{i+1},a_{i+2})}$ can be combined into a versal deformation $\xi_{X(a_i,a_{i+1},a_{i+2})}$ and so on, until we have a versal deformation $\xi_{X(a_2,\ldots,a_{e-1})}$.



Example

Example (Versal deformation of $X(a_2)$). *Consider the toric surface singularity $X(a_2)$ over a field k. If $\operatorname{char} k \nmid a_2$, then a versal deformation of $X(a_2)$ is given by*

$$\xi_{X(a_2)} = \begin{array}{ccc} X(a_2) & \longrightarrow & \operatorname{Spec} k[[t_1,\ldots,t_{a_2-1}]][X,Y,Z]/(XZ - Y^n + \sum_{i=0}^{n-2} t_{i+1}Y^i) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[[t_1,\ldots,t_{n-1}]]. \end{array}$$

On the other hand, if $\operatorname{char} k \mid a_2$, then a versal deformation is given by

$$\xi_{X(a_2)} = \begin{array}{ccc} X(a_2) & \longrightarrow & \operatorname{Spec} k[[t_1,\ldots,t_n]][X,Y,Z]/(XZ - Y^n + \sum_{i=0}^{n-1} t_{i+1}Y^i) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[[t_1,\ldots,t_n]]. \end{array}$$

Note: The versal deformation is dependent on the characteristic.

References

[Arn88] Jürgen Arndt. *Verselle Deformationen zyklischer Quotientensingularitäten*. PhD thesis, Staats-und Universitätsbibliothek Hamburg Carl von Ossietzky, 1988.

[KSB88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Inventiones mathematicae*, 91(2):299–338, Jun 1988.

[LMM] Christian Liedtke, Gebhard Martin, and Yuya Matsumoto. Linearly reductive quotient singularities. *to appear in Astérisque*.

[Rie74] Oswald Riemenschneider. Deformationen von quotientensingularitäten (nach zyklischen gruppen). *Mathematische Annalen*, 209:211–248, 09 1974.

[Sch71] Michael Schlessinger. Rigidity of quotient singularities. *Inventiones mathematicae*, 14(1):17–26, Mar 1971.