Singularities in Discrete-Time Fast-Slow Systems

Luca Arcidiacono, Supervisor: Prof. Dr. Christian Kühn

Department of Mathematics, Technical University of Munich

Abstract

To describe the dynamics of fast-slow systems near singularities, the continuation of the slow manifold beyond singularities is of particular interest. For the generic fold singularity, a result similar to the one obtained in the case of the continuous-time ODE system can be established for the discrete dynamical system, that arises out of the continuous one by a one-step method. Based on a graph transform method, the existence of a negatively invariant, attractive manifold is shown. It is piecewise given as a graph and in coordinates relative to the extension of the slow manifold.

Fast-Slow Systems

A system of ordinary differential equations is called a fast-slow system if it has one of the following structures, which can be transformed into each other by the change of time $\tau = \varepsilon t$.

$$\begin{aligned} x' &= \frac{dx}{d\tau} = f(x,y,\varepsilon) \\ \varepsilon y' &= \varepsilon\frac{dy}{d\tau} = g(x,y,\varepsilon) \end{aligned} \qquad (1) \qquad\qquad \begin{aligned} \dot{x} &= \frac{dx}{dt} = \varepsilon f(x,y,\varepsilon) \\ \dot{y} &= \frac{dy}{dt} = g(x,y,\varepsilon) \end{aligned} \qquad (2)$$

Here the small parameter $0 < \varepsilon \ll 1$ constitutes the ratio between the two time scales. We have $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$, the variable x is called the slow variable, y the fast variable.

Setting $\varepsilon = 0$ in (1) and (2) we obtain

$$\begin{aligned} x' &= f(x,y,0) \\ 0 &= g(x,y,0) \end{aligned} \qquad (3) \qquad\qquad \begin{aligned} \dot{x} &= 0 \\ \dot{y} &= g(x,y,0) \end{aligned} \qquad (4)$$

This limit case is also called singular limit as (3) is not a system of ODEs anymore, but a differential algebraic equation. The so-called slow flow or reduced flow of (3) is restricted to the set $C_0 := \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n \mid g(x,y,0) = 0\}$, which we refer to as the critical set, or if it is a manifold, as the critical manifold.

Since $\dot{x} = 0$, equations (4) can be seen as a dynamical system parametrized by the constant x. The set C_0 corresponds to the equilibria of (4).

Fenichel's Theorem and Singularities

A subset $S \subset C_0$ is called normally hyperbolic if the matrix $(D_y g)(p,0)$ has no eigenvalue with zero real part for all $p \in S$. The following theorem is an important tool to investigate the dynamics of the full system for $\varepsilon > 0$ in the vicinity of normally hyperbolic parts $S \subset C_0$.

Fenichel's Theorem

Suppose S_0 is a compact, normally hyperbolic submanifold (possibly with boundary) of the critical manifold C_0 of (1). Assume further that $f, g \in C^r, (r < \infty)$. Then for $\varepsilon > 0$ sufficiently small there exists a locally invariant manifold S_ε diffeomorphic to S_0, which is called a slow manifold. S_ε has Hausdorff distance $\mathcal{O}(\varepsilon)$ from S_0 and is C^r smooth. The flow on S_ε converges to the slow flow as $\varepsilon \to 0$, S_ε is normally hyperbolic and has the same stability properties as S_0. Usually S_ε is not unique but all manifolds satisfying the above lie at a Hausdorff distance $\mathcal{O}(e^{-K/\varepsilon})$ from each other for some $K > 0, K = \mathcal{O}(1)$.

This result is only applicable as long as the considered parts of the critical manifold are normally hyperbolic. Points where this property is lost are called singularities. A loss of normal hyperbolicity corresponds to a bifurcation of the parametrized equations (4).

The Blowup Technique

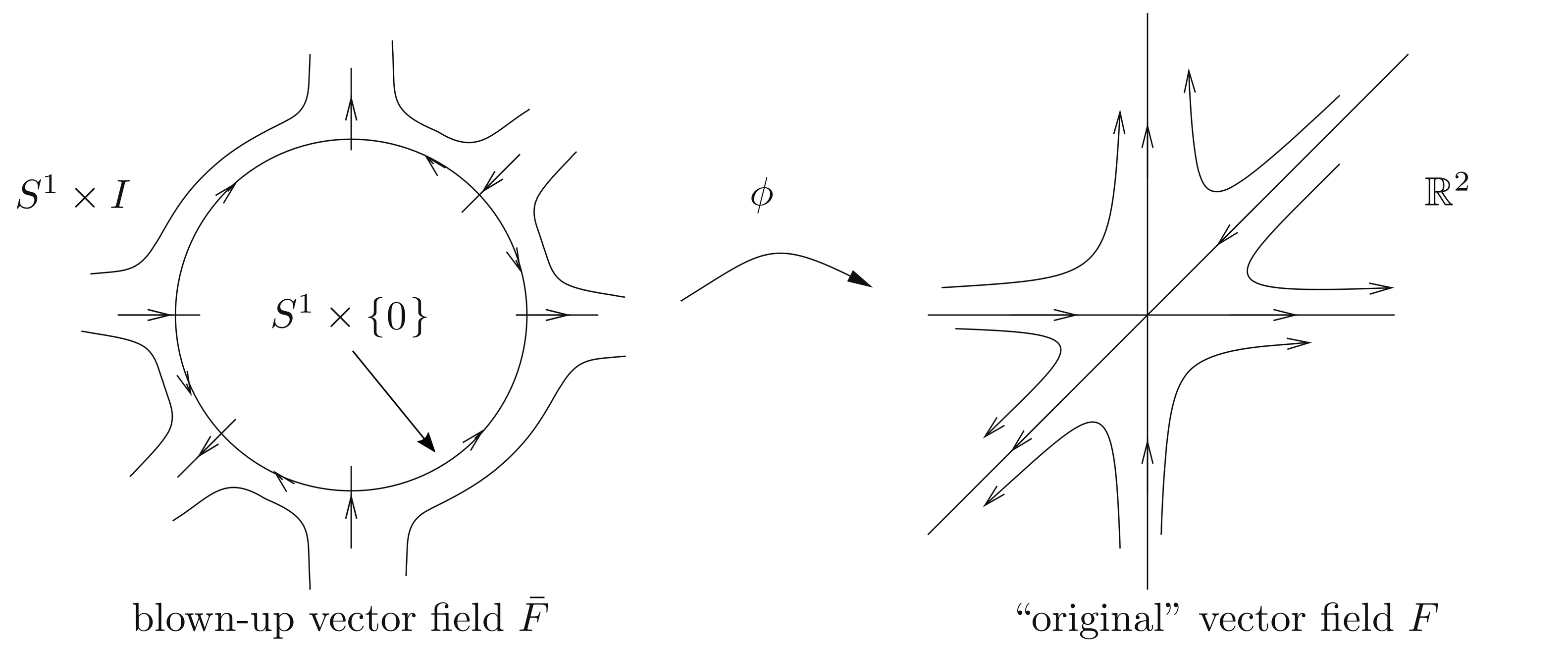
A method to gain insight into the dynamics around a non-hyperbolic equilibrium points, that also finds great application in the analysis of singularities, is the blowup technique.

Let $I \subseteq \mathbb{R}$ be an interval containing 0 and consider the map $\phi : S^{n-1} \times I \to \mathbb{R}^n$
$\phi(\bar{z}_1, \dots \bar{z}_n, r) = (r^{a_1}\bar{z}_1, \dots, r^{a_n}\bar{z}_n)$ for $(\bar{z}_1, \dots \bar{z}_n) \in S^{n-1}$, i.e. $\sum_{k=1}^n \bar{z}_k^2 = 1$

The quasihomogeneous blowup $\hat{F}$ of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with an equilibrium point at the origin is now defined by $\hat{F} = D\phi^{-1} \circ F \circ \phi$. The exponents $a_1, a_2, \dots, a_n \in \mathbb{N}$ are chosen depending on the quasihomogenicity of the underlying vector field F. The set $S^{n-1} \times \{0\}$ gets mapped onto the origin, i.e. the equilibrium is "blown up" to a sphere.

In general the blowup transform does not yet give any improvement, as there is no motion on the sphere. Under appropriate conditions we can further desingularize by scaling $\hat{F}$ and obtain the rescaled vector field $\bar{F} = \frac{1}{r^k}\hat{F}$, that allows to draw conclusions for the dynamics of the original, blown down system.

To simplify the calculations, we use suitable charts κ_i of the manifold $S^{n-1} \times I$ and describe the blowup by a coordinate transformation μ_i on the Euclidean space R^n, i.e. $\phi = \kappa_i \circ \mu_i$.



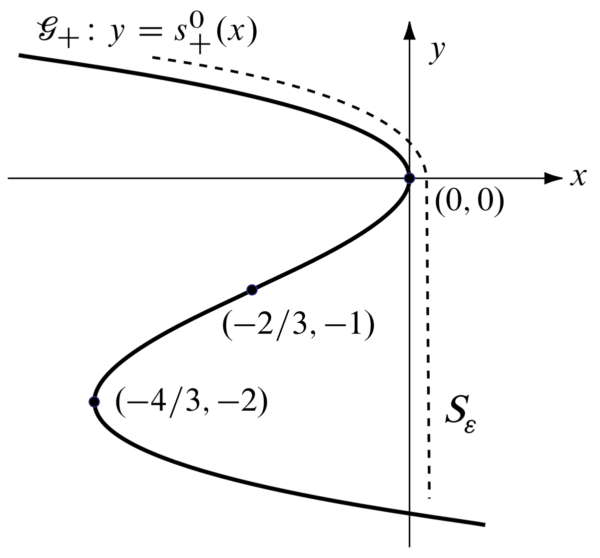
Blowup of the vector field $F(x,y) = (x^2 - 2xy,\ y^2 - 2xy)$, which has a non-hyperbolic equilibrium at $(0,0)$ with $a_1 = a_2 = 1$

The Fold Singularity

We consider the following fast-slow formulation of the van der Pol equation

$$\begin{aligned} \dot{x} &= \ \varepsilon^3 (1+y) &&=: \varepsilon^3 f(x,y) \\ \dot{y} &= -x - y^2 - \tfrac{y^3}{3} &&=: \ g(x,y) \end{aligned} \qquad (5)$$

The critical manifold $C_0 = \{x = -y^2 - \frac{y^3}{3}\}$ of (5) is a cubic curve. We calculate $D_y g(x,y) = -2y - y^2 = -y(y+2)$ so that C_0 is normally hyperbolic for $y \notin \{-2, 0\}$ and moreover attractive for $y \in \mathbb{R} \setminus [0,2]$. Apart from that we find a fold singularity (corresponding to the fold / saddle-node bifurcation) at the origin (as well as at the point $(x,y) = (-\frac{4}{3}, -2)$). The upper, normally hyperbolic branch $\mathcal{G}^+$ of the critical manifold can be described as the graph of a function $s_0^+ : (-\infty, 0) \to \mathbb{R}$. Away from the origin Fenichel's theorem yields a negatively invariant, attractive slow manifold S_ε close to $\mathcal{G}^+$, which is given as graph of some function s_1.



Near the origin we have $\dot{x} > 0$, so that the reduced flow is directed towards the singularity. It was shown that the extension of the slow manifold under the flow of (5) passes the fold point and then follows approximately a fast fiber in a distance of $\mathcal{O}(\varepsilon^2)$ to the negative y-axis. The quasihomogeneous blow up used to study the fold singularity has the form

$$x = r^2\bar{x} \quad y = r\bar{y} \quad \varepsilon = r\bar{\varepsilon} \qquad \text{for} \quad (\bar{x}, \bar{y}, \bar{z}, r) \in S^2 \times [0, R]$$

The Result in Discrete Time

We transfer the result of the continuous-time case into discrete time. Therefor we consider an iteration of the map $P : \mathbb{R}^4 \to \mathbb{R}^4$, that describes a one-step method with step size h.

$$P : \begin{pmatrix} x \\ y \\ \varepsilon \\ h \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\varepsilon} \\ \bar{h} \end{pmatrix} = \begin{pmatrix} x + h\varepsilon^3 f(x,y) + h^2\varepsilon^3 \hat{f}(x,y,\varepsilon,h) \\ y + hg(x,y) + h^2\hat{g}(x,y,\varepsilon,h) \\ \varepsilon \\ h \end{pmatrix} \qquad (6)$$

We suppose that this one-step method can be expressed by a Butcher series. This holds true for example for Runge-Kutta-methods or the explicit Euler method.

Main Result

There are $\delta_0, \nu_0 > 0$ such that for every $(\varepsilon, h) \in (0, \delta_0] \times (0, \nu_0]$ the map P admits an attractive, negatively invariant curve. The union of theses curves form a 3-dimensional attractive, negatively invariant manifold $\mathcal{M}$. The manifold $\mathcal{M}$ is described relatively to a reference manifold M^r – the slow manifold S_ε and its extension under the flow of (5) – which is negatively invariant and piecewise in charts $\Phi_1, \dots, \Phi_6$ defined as graph of smooth functions $s_1, \dots, s_6$. The manifold $\mathcal{M}$ is piecewise described as the graph of Lipschitz continuous functions $\sigma_1, \dots \sigma_6$, more precisely as $\mathcal{M}_i = \{(x,y) \mid x \in X_i,\ y = s_i(x,\varepsilon) + \sigma_i(x,\varepsilon,h)\}$ for $i = 1,2,3$ and $\mathcal{M}_i = \{(x,y) \mid y \in Y_i,\ x = s_i(y,\varepsilon) + \sigma_i(y,\varepsilon,h)\}$ for $i = 4,5,6$ for certain intervals X_i and Y_i.

The functions $\sigma_1, \dots \sigma_6$ satisfy $\sigma_i = \mathcal{O}(\varepsilon^3 h)$ for $i = 1,4$ and $\sigma_i = \mathcal{O}(\varepsilon^2 h)$ for $i = 2,3,5,6$.

Graph Transform

The proof of the main result uses the method of graph transform. A manifold $\mathcal{M}$ given in several charts $\Phi_i : \mathbb{R}^4 \to U_i \times V_i$ can be described as an element of the function space $\Sigma = \{\sigma = (\sigma_1, \dots, \sigma_6) \mid \sigma_i \in C_i(U_i, V_i)\}$. Applying the map P to $\mathcal{M}$ gives a set $P(\mathcal{M})$, under appropriate assumptions and under an appropriate choice of the spaces $C_i(U_i, V_i)$ there is a set $\mathcal{M}^* \subset P(\mathcal{M})$ which is again described by an element $\sigma^* \in \Sigma$. Thereby the mapping P induces an operator $\mathcal{F} : \Sigma \to \Sigma$. For the proof we have to assure that the map P flows from one chart to the next one and show that $\mathcal{F}$ is a contraction.

References

1. Christian Kuehn. Multiple Time Scale Dynamics, volume 191. Springer International Publishing, Cham, 2015.
2. Kaspar Nipp and Daniel Stoffer. Invariant manifolds in discrete and continuous dynamical systems, volume 21 of EMS Tracts in mathematics. European Mathematical Society, Zürich, 2013.