Uniqueness and Non-Uniqueness for Spin-Glass Ground States on Trees

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Abstract

We consider a Spin Glass at temperature $T = 0$ where the underlying graph is a locally finite tree. We prove for a wide range of coupling distributions that uniqueness of ground states is equivalent to the maximal flow from any vertex to ∞ (where each edge e has capacity $|J_e|$) being equal to zero, which is equivalent to recurrence of the simple random walk on the tree. Furthermore we give a sufficient condition for the above statements.

Spin Glasses

Let $G = (V, E)$ be a finite graph; For i.i.d. absolutely continuous random variables $(J_e)_{e\in E}$ and $\sigma \in \{-1, +1\}^V$ define the Hamiltonian of the system as

$$\mathcal{H}(\sigma) := -\sum_{\{x,y\}\in E} J_{xy}\sigma_x\sigma_y \qquad (1)$$

The random variables can take both positive and negative values; The Hamiltonian is clearly invariant under a global spin flip. For such a Hamiltonian and an inverse temperature $\beta > 0$ one can define the Gibbs distribution on $\{-1, +1\}^V$ by

$$G_\beta(\sigma) := \frac{1}{Z}e^{-\beta\mathcal{H}(\sigma)} \qquad (2)$$

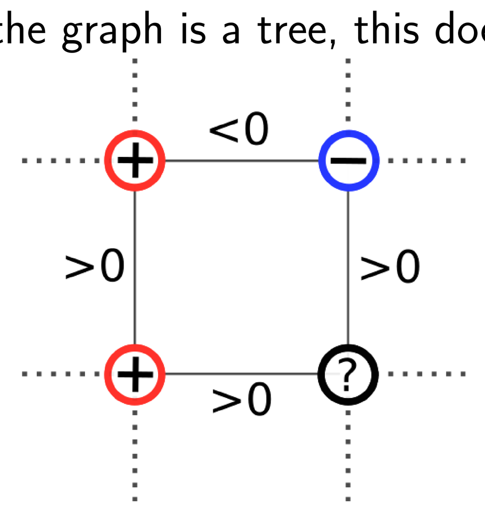
where Z is a normalizing constant, depending on the random variables and β. When temperature approaches zero - or equivalently $\beta \to \infty$ - there is more mass on configurations with low Hamiltonian, and indeed

$$G_\beta \to \frac{1}{2}\delta_{\{\sigma\}} + \frac{1}{2}\delta_{\{-\sigma\}} \qquad (3)$$

where σ and $-\sigma$ are the minimizers of $\mathcal{H}(\sigma)$, the **Ground states**. There is an equivalent characterization of ground states, which can also be extended to infinite graphs: σ is a ground state if and only if

$$\sum_{\{x,y\}\in\partial B} J_{xy}\sigma_x\sigma_y \geq 0 \;\forall\; B \subset V \qquad (4)$$

where $\partial B \subset E$ is the set of edges with exactly one end in B. Some edge $e = \{x, y\}$ is called **satisfied** if $J_{xy}\sigma_x\sigma_y \geq 0$. For most interesting graphs it is not possible, that every edge is satisfied, see the example below. This phenomenon is called frustration, if the graph does not have any loops, i.e. if the graph is a tree, this does not occur.

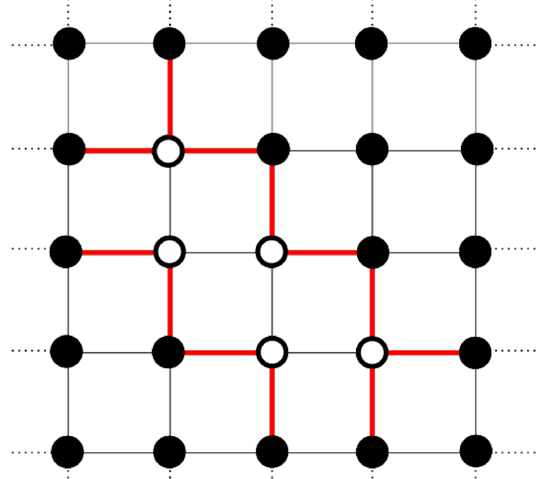


Ground States for infinite Graphs

As mentioned above, for an infinite, but still locally finite graph $G = (V, E)$ one can define ground states $\sigma \in \{-1, +1\}^V$ by the identity

$$\sum_{\{x,y\}\in\partial B} J_{xy}\sigma_x\sigma_y \geq 0 \;\forall\; B \subset V \; finite \qquad (5)$$

B are the empty vertices and ∂B the red edges in the figure below



Let $\mathcal{G}(J) := \left\{\sigma \in \{-1, +1\}^V : \sigma \text{ is a ground state}\right\}$. One of the central questions in Spin-Glass theory and the motivation of this Thesis is to determine $|\mathcal{G}(J)|$ for different graphs.

Ground states for infinite trees

For trees, there are two **natural ground states**, the ones where every edge is satisfied. However, there still might be more than these two. The thesis provides for a wide range of coupling distributions four equivalent statements for uniqueness and non-uniqueness of the ground states:

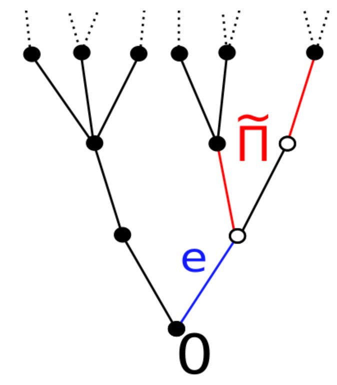
Let $T = (V, E)$ be a tree and suppose $(J_e)_{e\in E}$ are i.i.d. with $\mathbb{P}(J_e \in (-\epsilon, \epsilon)) = \Theta(\epsilon)$. Then the following are equivalent:

i) The natural ground states are the only ground states a.s.
ii) $\inf\{\sum_{e\in\Pi} |J_e| \;:\; \Pi$ cutset separating 0 and $\infty\} = 0$ a.s.
iii) MaxFlow$(0 \to \infty, \langle|J_e|\rangle) = 0$ a.s.
iv) The simple random walk on T = (V,E) is recurrent

Ideas of the proof

The equivalence of $ii)$ and $iii)$ is obtained by the MaxFlow-MinCut-Theorem, see [2] and [5]

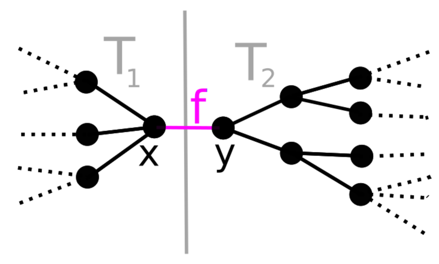
$ii) \Rightarrow i)$: Let σ be a ground state, for every edge e we can almost surely find some set $\tilde{\Pi}$ lying in the tree above e - see the figure below - and satisfying $\sum_{f\in\tilde{\Pi}} |J_f| < |J_e|$



From (4) it is not hard to see that e has to be satisfied. As e was arbitrary, the natural ground states are the only ground states.

$i) \Rightarrow iii)$: If $MaxFlow(0 \to \infty, \langle|J_e|\rangle) > 0$ one can show the existence of two subtrees T_1, T_2 and an edge $f = \{x, y\}$ connecting these subtrees (see the figure below) such that

$$MaxFlow_{T_1}(x \to \infty, \langle|J_e|\rangle) > |J_f| \qquad (6)$$
$$MaxFlow_{T_2}(y \to \infty, \langle|J_e|\rangle) > |J_f| \qquad (7)$$



The configuration, where f is the only non satisfied edge is a ground state. Furthermore one can even show the existence of infinitely many such edges f, so $|\mathcal{G}(J)| = \infty$ in this case.

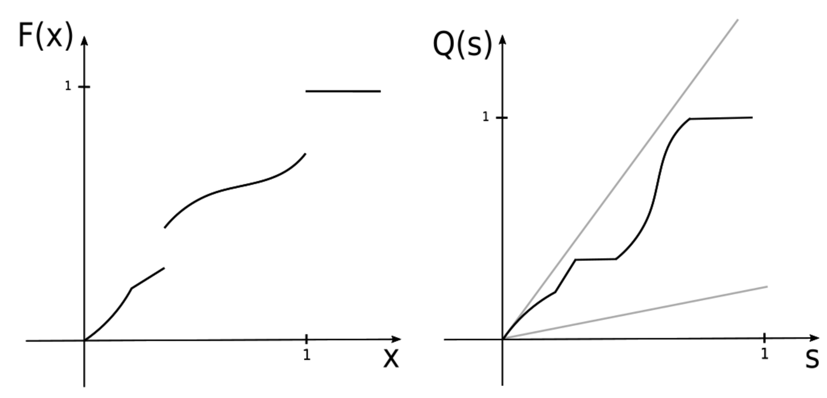
$iii) \Leftrightarrow iv)$: The main tool in proving this equivalence is the following Theorem, see [4]:
Let $(\kappa_e)_{e\in E}$ be independent exponentially distributed with mean c_e, T is a tree and Z its leaves. Then

$$\mathbb{E}\left[MaxFlow(0 \to Z, \langle\kappa_e\rangle)\right] \geq Conduct(0 \to Z, \langle c_e\rangle) \qquad (8)$$
$$\mathbb{E}[MaxFlow(0 \to Z, \langle\kappa_e\rangle)] \leq 2Conduct(0 \to Z, \langle c_e\rangle) \qquad (9)$$

Corollary

Let $(\kappa_e)_{e\in E}$ be i.i.d. exponentials with mean 1, then $MaxFlow(0 \to \infty, \langle\kappa_e\rangle) = 0$ a.s. if and only if the simple random walk is recurrent.

To extend this corollary for other coupling distributions one uses positive homogenity and monotonicity of the Maximal flow. Furthermore note that MaxFlow$(0 \to \infty, \langle\kappa_e\rangle) > 0$ if and only if MaxFlow$(0 \to \infty, \langle\kappa_e \wedge 1\rangle) > 0$. As $(J_e)_{e\in E}$ are distributed according to some distribution of linear growth we can find - using the quantile function, see the figure below - some coupling of J_e and I_e, where $(I_e)_{e\in E}$ are i.i.d. Unif(0,1), such that $c \cdot I_e \leq J_e \leq C \cdot I_e \;\forall e \in E$ and $0 < c < C < \infty$



by applying this idea first to J_e and then to κ_e we get the desired statement. Note that we really need the property $\mathbb{P}(J_e \in (-\epsilon, \epsilon)) = \Theta(\epsilon)$ here.

References

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