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Quantum Channels and Infinitesimal Divisibility

Matthias C. Caro

Center for Mathematical Sciences, Technische Universität München
Chair of Mathematical Physics

Technische Universität München
Elitenetzwerk Bayern
Universität Augsburg University

Abstract

The time evolution of a general quantum system is given by a so-called quantum channel. In contrast to the time evolution of a closed quantum system which is governed by the Schrödinger equation, not every quantum channel is an element of a semigroup of linear maps. We present the results of Gorini, Kossakowski Sudarshan and Lindblad that characterize semigroups of quantum channels via their generators. Then we discuss the approach of Wolf and Cirac to introduce a more general class of channels that has a structure similar to the semigroup structure but is more general, the class of infinitesimal divisible channels. Finally, we sketch a possible approach to obtain necessary criteria for infinitesimal divisibility.

Introduction and basic framework

Quantum theory entails many phenomena that are counterintuitive when compared with our everyday experience. It is therefore all the more important to find a suitable mathematical description. One such framework is presented in the following with emphasis on the time evolution of a system. Thereafter, a certain kind of time evolution is discussed in greater detail and a characterization of a more general class of evolutions is given.

Let $\mathcal{M}_d$ be the complex $d \times d$-matrices. We define the state space to be $\mathcal{S}(\mathbb{C}^d) = \{\rho \in \mathcal{M}_d | \rho \geq 0, \mathrm{tr}[\rho] = 1\}$. With this definition $\mathrm{tr}[\rho \cdot] : \mathcal{E}(\mathbb{C}^d) \to [0,1]$ becomes something like a probability measure on the set of effects $\mathcal{E}(\mathbb{C}^d) := \{E \in \mathcal{M}_d | 0 \leq E \leq Id\}$, it gives the probability of measuring an effect E in the state ρ. (The familiar name for this connection of states and effects with measurement probabilities is Born rule.) This relation in particular shows that we can think of a time evolution either as an evolution of states or as an evolution of effects, namely: if T is the evolution on $\mathcal{S}(\mathbb{C}^d)$, then the evolution T^* on $\mathcal{E}(\mathbb{C}^d)$ has to satisfy $\mathrm{tr}[T(\rho)E] = \mathrm{tr}[\rho T^*(E)]\ \forall \rho \in \mathcal{S}(\mathbb{C}^d),\ E \in \mathcal{E}(\mathbb{C}^d)$.

Quantum channels and their descriptions

We first define the notion of a quantum channel in an axiomatic way.
Definition 1: A map $T : \mathcal{M}_d \to \mathcal{M}_d$ is called (quantum) operation if

1. T is linear
2. T is completely positive, i.e. $T \otimes Id : \mathcal{M}_d \otimes \mathcal{M}_n \to \mathcal{M}_d \otimes \mathcal{M}_n$ is positive $\forall n \in \mathbb{N}$
3. T is trace non-increasing, i.e. $tr[T(\rho)] \leq tr[\rho]\ \forall \rho \in \mathcal{M}_d$.

If T is even trace preserving, then T is called (quantum) channel. We define $\mathcal{T}_d$ to be the set of quantum channels over $\mathcal{M}_d$.

Example 1: We define the partial trace $\mathrm{tr}_B : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ to be the linear map satisfying $\mathrm{tr}[\mathrm{tr}_B[T]C] = \mathrm{tr}[T(C \otimes Id_B)]\ \forall T \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\forall C \in \mathcal{B}(\mathcal{H}_A)$. One can show that this is a channel according to the definition above. Physically, the partial trace tr_B means that we discard the system B in a way that is consistent with measurement statistics. In particular, the partial trace tr_B maps $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ into $\mathcal{S}(\mathcal{H}_A)$.

Stinespring's dilation theorem shows that Definition 1 agrees with the intuition of a physicist, which is not obvious from the definition itself.
Theorem 1: (Stinespring) A linear map $T : \mathcal{M}_d \to \mathcal{M}_d$ is a quantum channel iff there exist a Hilbert space $\mathcal{H}_E$, a pure state $\xi \in \mathcal{S}(\mathcal{H}_E)$ and a unitary U acting on $\mathbb{C}^d \otimes \mathcal{H}_E$ such that $\forall \rho \in \mathcal{S}(\mathbb{C}^d)$: $T(\rho) = \mathrm{tr}_E[U(\rho \otimes \xi)U^\dagger]$.

So we can embed the considered open system into a larger closed system where the unitary dynamics is given by a Schrödinger equation. If we then look only at the open system as a subsystem, we obtain the evolution and it is given by a quantum channel.

Two further useful characterizations can be obtained by the operator-sum form called Kraus representation (which can be derived using Stinespring's theorem) and the Choi-Jamiolkowski isomorphism, also often referred to as channel-state duality. Both are used in many proofs in the context of quantum channels.
Theorem 2: (Kraus) A linear map $T : \mathcal{M}_d \to \mathcal{M}_d$ is a quantum channel if and only if there exist finitely many matrices $\{K_i\}_i$ in $\mathcal{M}_d$ such that

$$T(\rho) = \sum_n K_n \rho K_n^\dagger, \quad \sum_n K_n^\dagger K_n = Id.$$

Theorem 3: (Choi-Jamiolkowski) Let $\Omega = \frac{1}{\sqrt{d}} \sum_i |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Then the linear map $\mathcal{J} : \mathcal{B}(\mathcal{M}_d) \to \mathcal{M}_{d^2}$ given by $\mathcal{J}(T) = \tau := (T \otimes Id_d)(|\Omega\rangle\langle\Omega|)$ defines an isomorphism, the so-called Choi-Jamiolkowski isomorphism.
The inverse is given by the relation $\mathrm{tr}[AT(B)] = d\,\mathrm{tr}[\tau(A \otimes B^T)]\ \forall A, B \in \mathcal{M}_d$.

References (Excerpt)

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Quantum dynamical semigroups

Definition 2: A family of linear maps $T_t : \mathcal{M}_d \to \mathcal{M}_d$ with time parameter $t \in \mathbb{R}_+$ is called dynamical semigroup if $\forall t, s \in \mathbb{R}_+ : T_t T_s = T_{t+s}$ and $T_0 = Id$. If in addition the map $t \mapsto T_t$ is continuous (we are working on finite dimensional spaces, so there is no need to specify the norm here), then the family is called a continuous dynamical semigroup.

A well-known statement from the theory of semigroups tells us that a strongly continuous semigroup of operators has a generator. If we consider finite-dimensional spaces, then this result becomes the following:
Proposition 1: Let $\{T_t\}_{t \in \mathbb{R}_+}$ be a continuous dynamical semigroup. Then the map $t \mapsto T_t$ is differentiable and $T_t = e^{tL}$ for some linear $L : \mathcal{M}_d \to \mathcal{M}_d$.

Many properties of semigroups have a counterpart in certain properties of its generator. In 1976, G. Lindblad and Gorini/Kossakowski/Sudarshan independently reached a result that gives a complete characterization of the generators of continuous semigroups of quantum channels.
Theorem 4: (GKLS) A linear map $L : \mathcal{M}_d \to \mathcal{M}_d$ is the generator of a dynamical semigroup of quantum channels if and only if it can be written in one of the following equivalent forms:

$$\begin{aligned} L(\rho) &= \phi(\rho) - \kappa\rho - \rho\kappa^\dagger \\ &= i[\rho, H] + \sum_j L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\} \\ &= i[\rho, H] + \sum_{j,k=1}^{d^2-1} C_{j,k}([F_j, \rho F_k^\dagger] + [F_j \rho, F_k^\dagger]) \end{aligned}$$

where ϕ is a completely positive linear map satisfying $\phi^*(Id) = \kappa + \kappa^\dagger$, $\kappa \in \mathcal{M}_d$, $H = H^\dagger \in \mathcal{M}_d$ selfadjoint, $\{L_j\}_j$ a set of matrices in $\mathcal{M}_d$, $C \in \mathcal{M}_{d^2-1}, C \geq 0$ and $\{F_j\}_{j=1,\ldots,d^2-1}$ an orthonormal basis of the space $\{A \in \mathcal{M}_d :\ tr[A] = 0\}$.

The following Lemma shows that every channel gives rise to a quantum dynamical semigroup that approximates the channel.
Lemma 1: Let $T : \mathcal{M}_d \to \mathcal{M}_d$ be a channel. Then $\{e^{t(T-id)}\}_{t \geq 0}$ is a completely positive semigroup. If we choose $U_0 \in \mathcal{T}_d, U_0(\rho) = U_0 \rho U_0^\dagger$ to be the unitary channel satisfying $\mathrm{tr}_{\mathcal{H}}[TU_0] = \sup_{U \in \mathcal{T}_d \text{ unitary}} \mathrm{tr}_{\mathcal{H}}[TU]$, then $TU_0 - id$ is the Lindblad generator of a purely dissipative semigroup.

Infinitesimal divisibility of channels

A quantum dynamical semigroup $\{T_t\}_{t \geq 0}$ with generator L satisfies the differential equation $\frac{d}{dt}T_t = LT_t$. The following definition is motivated by looking at this differential equation and allowing a time dependent generator.
Definition 3: Let $t > 0$ and $T : [0,t] \times [0,t] \to \mathcal{T}_d$ be a continuous map satisfying

1. $T(t_3, t_2)T(t_2, t_1) = T(t_3, t_1)\ \forall\ 0 \leq t_1 \leq t_2 \leq t_3 \leq t$
2. $\lim_{\epsilon \to 0} \|T(\tau + \epsilon, \tau) - Id\| = 0\ \forall \tau \in [0, t)$.

Then we call the family $\{T(t_2, t_1)\}_{0 \leq t_1 \leq t_2 \leq t}$ a continuous completely positive evolution. Moreover we define $\mathcal{J}$ to be the set of channels that are contained in a continuous completely positive evolution.

Intuitively, this definition is very similar to the following:
Definition 4: We define the set $\mathcal{I} = \{T \in \mathcal{T}_d |\ \forall \epsilon > 0\ \exists n \in \mathbb{N},\ \{T_i\}_{1 \leq i \leq n} \subseteq \mathcal{T}$ such that (i) $\|T_i - Id\| \leq \epsilon$ and (ii) $\prod_{i=1}^n T_i = T\}$.
We call $\overline{\mathcal{I}}$ the set of infinitesimal divisible channels. We call $\mathcal{I}'$ the set of channels that satisfy the defining condition of $\mathcal{I}$ with approximations of the form $T_i = e^{L_i}$ for Lindblad generators L_i.

Though all these definitions seem very similarly intuitively, only the inclusions $\mathcal{I}' \subset \mathcal{J} \subset \mathcal{I}$ can easily be seen. The following theorem by Wolf/Cirac establishes the equivalence of the definitions. (The proof uses an approximation based on Lemma 1.)
Theorem 5: With the definitions above we have $\overline{\mathcal{I}'} = \overline{\mathcal{J}} = \overline{\mathcal{I}}$.

It can easily be seen from multiplicativity and continuity of the determinant that every infinitesimal divisible channel T satisfies $\det(T) \geq 0$. The structure of the Lindbladians that can be used to approximate T now suggests that it is possible to prove inequalities concerning eigenvalues or singular values (and hence the determinant) of T using the Lie-Trotter formula. In the case of normal Lindblad operators one can show that this hope is indeed justified. However, it seems difficult to generalize such results to arbitrary Lindblad generators and thus to general infinitesimal divisible channels.