On the Variational Analysis of a Chirality Transition Model in $\mathbb{R}^2$

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Abstract

In the thesis, we investigate a minimal energy model for configurations of spins lying in the sphere $\mathbb{S}^1$ on a two-dimensional lattice with competing nearest-neighbour (NN) and next-nearest-neighbour (NNN) interactions. We prove, that, depending on the interaction parameters, the ground states are ferromagnetic or helimagnetic. In the case of a helimagnetic ground state, the system is characterized by its horizontal and its vertical chirality, both being either clockwise or counterclockwise. The goal of the thesis is to analyze the emergence of chirality transitions in spin configurations with small but not minimal energy, when the number of particles is large and the parameters are close to the ferromagnet/helimagnet transition point. Building on a similar analysis for one-dimensional lattices, which has been peformed in [2], this is done by investigating the Γ-limit of the energy as the number of particles diverges and the parameters approach the ferromagnet/helimagnet transition point. Suitably renormalizing the energies, we find out, that the corresponding Γ-limit allows only very special chirality transitions.

The Energy Model

Let $Q=[0,1]^2$ denote the two-dimensional unit square and $\lambda_n^{\text{hor}}, \lambda_n^{\text{vert}} \to 0$ be two positive vanishing sequences of horizontal and vertical lattice spacings.
A **spin configuration** on the lattice $\left(\lambda_n^{\text{hor}}\mathbb{Z} \times \lambda_n^{\text{vert}}\mathbb{Z}\right) \cap Q$ is a function

$$u : \left(\lambda_n^{\text{hor}}\mathbb{Z} \times \lambda_n^{\text{vert}}\mathbb{Z}\right) \cap Q \to \mathbb{S}^1,\ (\lambda_n^{\text{hor}} i, \lambda_n^{\text{vert}} j) \mapsto u[i,j].$$

Energy of the configuration u:

$$E_n(u) = \lambda_n^{\text{hor}}\lambda_n^{\text{vert}} \sum_{\substack{i<1/\lambda_n^{\text{hor}}-2 \\ j<1/\lambda_n^{\text{vert}}-2}} \underbrace{-\alpha^{\text{hor}}\langle u[i,j], u[i+1,j]\rangle}_{\text{ferromagnetic horizontal NN interactions}} + \underbrace{\langle u[i,j], u[i+2,j]\rangle}_{\text{antiferromagnetic horizontal NNN interactions}}$$
$$+ \underbrace{-\alpha^{\text{vert}}\langle u[i,j], u[i,j+1]\rangle}_{\text{ferromagnetic vertical NN interactions}} + \underbrace{\langle u[i,j], u[i,j+2]\rangle}_{\text{antiferromagnetic vertical NNN interactions}};$$

$\alpha^{\text{hor}}, \alpha^{\text{vert}} > 0$: Horizontal and vertical NN interaction parameters, may depend on n.
Boundary conditions: We require a special kind of periodic boundary conditions for the configuration u and redefine $E_n(u) := +\infty$ for all u, which do not satisfy these conditions.
Embedding into a common function space: We embed the set of all possible configurations into the space $L^\infty(Q)$ by associating each configuration u with the corresponding L^∞-function, which is constant on every cell

$$\left(\left[\lambda_n^{\text{hor}} i, \lambda_n^{\text{hor}}(i+1)\right) \times \left[\lambda_n^{\text{vert}} j, \lambda_n^{\text{vert}}(j+1)\right)\right) \cap Q$$

of the lattice. We then extend each energy E_n with the value $+\infty$ to the whole $L^\infty(Q)$.

Ground States of the System

Theorem 1:
Assume that $\lambda_n^{\text{hor}}, \lambda_n^{\text{vert}} < \frac{1}{4}$. Then, a spin configuration u is a minimizer of E_n if and only if there exist $\chi^{\text{hor}}, \chi^{\text{vert}} \in \{-1,+1\}$ and $\varphi_0 \in \mathbb{R}$, such that for all (i,j) with $i < 1/\lambda_n^{\text{hor}} - 2$ or $j < 1/\lambda_n^{\text{vert}} - 2$ we have

$$u[i,j] = (\cos(\varphi_{i,j}), \sin(\varphi_{i,j})),$$

where

$$\varphi_{i,j} = \varphi_0 + \chi^{\text{hor}} \cdot i \cdot \varphi^{\text{hor}} + \chi^{\text{vert}} \cdot j \cdot \varphi^{\text{vert}}$$

and

$$\text{for dir} = \text{hor, vert}: \quad \varphi^{\text{dir}} = \begin{cases} \arccos\left(\frac{\alpha^{\text{dir}}}{4}\right) & \text{if } \alpha^{\text{dir}} \le 4 \\ 0 & \text{otherwise} \end{cases}.$$

Case $\alpha^{\text{hor}}, \alpha^{\text{vert}} \ge 4$: The ground states are uniform (ferromagnetic).
Case $\alpha^{\text{hor}}, \alpha^{\text{vert}} < 4$: On each row of the lattice, the spins "rotate" clockwise or counterclockwise, depending on the horizontal chirality χ^{hor} by a fixed angle φ^{hor}. On each column, the spins rotate by the angle φ^{vert} with a vertical chirality χ^{vert}.

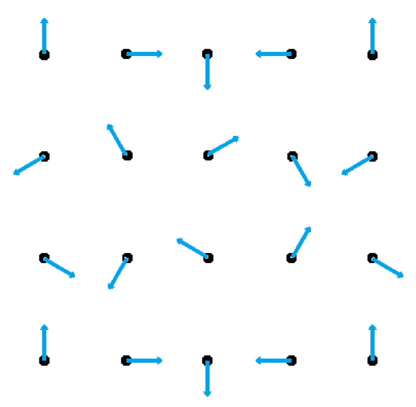

Fig. 1: Ground state for $\varphi^{\text{hor}} = \pi/2$, $\varphi^{\text{vert}} = 2\pi/3$, $\chi^{\text{hor}} = -1$ and $\chi^{\text{vert}} = 1$.

Zero Order Γ-Convergence

An application of the general theory about the asymptotic behaviour of pairwise interacting, discrete energies, presented in [1], yields the following Γ-convergence result:
Theorem 2:
Suppose, that $\lambda_n^{\text{hor}} = \lambda_n^{\text{vert}}$ for all n and let $\alpha^{\text{hor}} = \alpha_n^{\text{hor}}$ and $\alpha^{\text{vert}} = \alpha_n^{\text{vert}}$ depend on n and converge to 4 from below as $n \to \infty$. Then, the Γ-limit of E_n with respect to the weak*-convergence in L^∞ is given by

$$\Gamma\text{-}\lim_{n\to\infty} E_n(u) = \begin{cases} -6 & \text{if } u \in L^\infty(Q, \overline{B}_1(0)) \\ +\infty & \text{otherwise} \end{cases}.$$

Renormalization

Set $\alpha_n^{\text{hor}} = 4(1-\delta_n^{\text{hor}})$, $\alpha_n^{\text{vert}} = 4(1-\delta_n^{\text{vert}})$ for two positive sequences $\delta_n^{\text{hor}}, \delta_n^{\text{vert}} \to 0$.
Goal: Find a Γ-limit of the renormalized energies

$$\overline{H}_n(w,z) = \frac{E_n(u) - \min E_n}{\sqrt{2}\lambda_n^{\text{hor}}(\delta_n^{\text{hor}})^{3/2}}$$

with respect to the L^1-convergence of the order parameters w, z, defined by

$$w[i,j] := \sqrt{\frac{2}{\delta_n^{\text{hor}}}} \sin\left(\frac{1}{2}\Theta(u[i,j], u[i+1,j])\right) \approx \frac{1}{\sqrt{2\delta_n^{\text{hor}}}}\Theta(u[i,j], u[i+1,j]) \text{ and}$$

$$z[i,j] := \sqrt{\frac{2}{\delta_n^{\text{vert}}}} \sin\left(\frac{1}{2}\Theta(u[i,j], u[i,j+1])\right) \approx \frac{1}{\sqrt{2\delta_n^{\text{vert}}}}\Theta(u[i,j], u[i,j+1]);$$

$\Theta(v,w) \in [-\pi, \pi)$: Oriented angle between the vectors $v, w \in \mathbb{S}^1$.
w, z represent the horizontal and vertical chirality of the configuration u.

The Expected Result

We can approximately rewrite $\overline{H}_n$ as the discrete counterpart of the energy

$$G_n(w,z) = \int_0^1 F_{\lambda_n^{\text{hor}}/\sqrt{2\delta_n^{\text{hor}}}}(w(\cdot, x_2))\,\mathrm{d}x_2 + \frac{\lambda_n^{\text{vert}}(\delta_n^{\text{vert}})^{3/2}}{\lambda_n^{\text{hor}}(\delta_n^{\text{hor}})^{3/2}} \int_0^1 F_{\lambda_n^{\text{vert}}/\sqrt{2\delta_n^{\text{vert}}}}(z(x_1, \cdot))\,\mathrm{d}x_1,$$

where

$$F_\varepsilon(f) = \int_0^1 \frac{1}{\varepsilon}\left(f(x)^2 - 1\right)^2 + \varepsilon f'(x)^2\,\mathrm{d}x \qquad \left(f \in W^{1,2}(0,1)\right)$$

is the usual one-dimensional Modica-Mortola functional.

$$\text{If } \frac{\lambda_n^{\text{hor}}}{\sqrt{2\delta_n^{\text{hor}}}}, \frac{\lambda_n^{\text{vert}}}{\sqrt{2\delta_n^{\text{vert}}}} \to 0 \text{ and } \frac{\lambda_n^{\text{hor}}}{\lambda_n^{\text{vert}}} \to r^\lambda \in (0,\infty),\ \frac{\delta_n^{\text{hor}}}{\delta_n^{\text{vert}}} \to r^\delta \in (0,\infty),$$

we expect:
$\Gamma\text{-}\lim_n \overline{H}_n(w,z)$ is finite only if $w(\cdot, x), z(x, \cdot) \in BV((0,1), \{\pm 1\})$ for a.e. $x \in (0,1)$ and

$$\Gamma\text{-}\lim_{n\to\infty} \overline{H}_n(w,z) = \frac{8}{3}\left(\int_0^1 \#S(w(\cdot, x_2))\,\mathrm{d}x_2 + \frac{1}{r^\lambda (r^\delta)^{3/2}} \int_0^1 \#S(z(x_1, \cdot))\,\mathrm{d}x_1\right).$$

Because of the construction of w, z we expect:
$\Gamma\text{-}\lim_n \overline{H}_n(w,z)$ is only finite, if w, z satisfy the differential constraint $\partial_2 w = r^\lambda/\sqrt{r^\delta}\ \partial_1 z$ in the sense of distributions.

Results Proved in the Thesis

1. From the expected result, the liminf inequality holds true.
2. If $w, z \in BV(\mathring{Q})$, then the conditions $w, z \in \{\pm 1\}$ a.e. and $\partial_2 w = r^\lambda/\sqrt{r^\delta}\ \partial_1 z$ together imply very strict requirements on the normals ν_w and ν_z.

Example: w **and** z **jump at most along one smooth line**
$\Longrightarrow$ there are only a few possibilities:

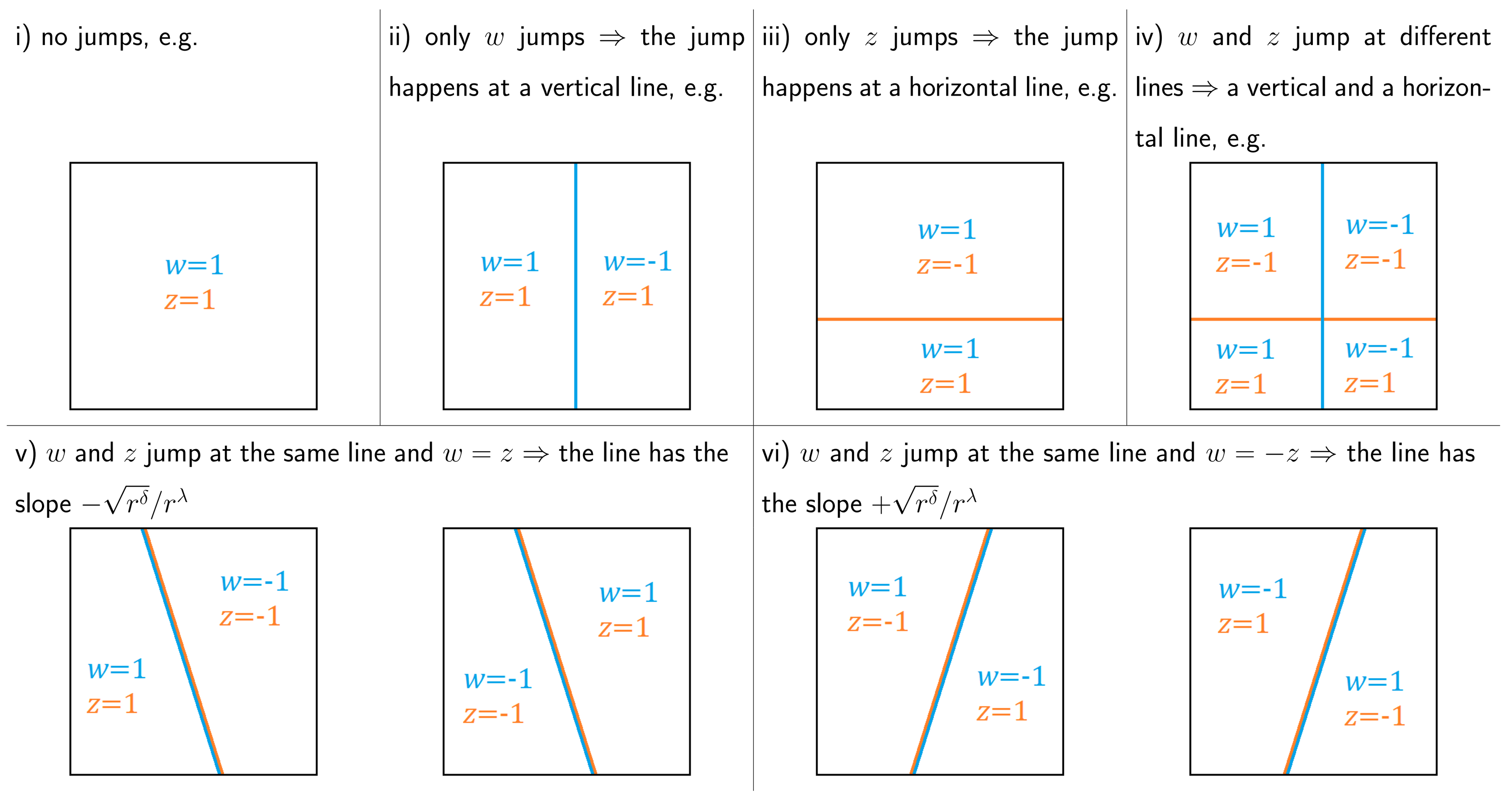

Fig. 2: Possible configurations for w and z with at most one jump. In blue the jump lines of w and in orange the jump lines of z.

3. From the expected result, the limsup inequality holds true in the cases, where w, z are of the form as in i) – iv).

References

[1] R. Alicandro, M. Cicalese, A. Gloria, Variational description of bulk energies for bounded and unbounded spin systems, *Nonlinearity* **21** (2008), no. 8, 1881–1910.
[2] M. Cicalese, F. Solombrino, Frustrated ferromagnetic spin chains: a variational approach to chirality transitions, *Journal of Nonlinear Science* **25** (2015), no. 2, 219–313.