Sheaves on the subanalytic site and tempered solutions of $\mathcal{D}$-modules on curves

Andreas Hohl, Supervisor: Prof. Dr. Marco Hien

Institut für Mathematik, Universität Augsburg

Lehrstuhl für Algebra und Zahlentheorie

TopMath
Mathematik mit Promotion

Elitenetzwerk Bayern

Abstract

The theory of $\mathcal{D}$-modules is an algebraic approach to linear partial differential equations. A $\mathcal{D}$-module can represent a system of differential equations and one can define the notion of *solutions* in a given function space. In general, different (i.e. not isomorphic) $\mathcal{D}$-modules can have the same holomorphic solutions. The concept of *tempered solutions* refines this notion and enables us to better distinguish $\mathcal{D}$-modules by their solutions in certain cases. The aim of the thesis was to introduce tempered holomorphic functions and work through an explicit example in the one-dimensional case. However, tempered functions do not form a sheaf on the usual topology of a complex manifold. For this reason, one has to work on the *subanalytic site*.

$\mathcal{D}$-modules and differential equations

On a complex manifold, we have the *sheaf of linear partial differential operators with holomorphic coefficients* $\mathcal{D}_X$. It is a sheaf of non-commutative $\mathbb{C}$-algebras.
Locally, an operator $P \in \mathcal{D}_X(U)$ is of the form

$$\sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq m}} a_\alpha(z) \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$$

for $m \in \mathbb{N}$ and holomorphic functions a_α.

To an operator P, globally defined on X, one associates the $\mathcal{D}_X$-module
$\mathcal{M}_P := \mathcal{D}_X/(\mathcal{D}_X P)$.
The holomorphic solutions of the equation $Pu = 0$ are then represented by

$$\mathscr{S}(\mathcal{M}_P) := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_P, \mathcal{F})$$

because

$$\mathscr{S}(\mathcal{M}_P)(U) \cong \{u \in \mathcal{F}(U) \,|\, Pu = 0\}.$$

More generally, if $\mathcal{M}$ is a $\mathcal{D}_X$-module and $\mathcal{F}$ is a $\mathcal{D}_X$-module of function spaces on X (e. g. $\mathcal{F} = \mathcal{O}_X$ or $\mathcal{F} = \mathcal{C}_X^\infty$), one calls

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$$

the **solution complex** of $\mathcal{M}$ with values in $\mathcal{F}$.

The subanalytic site

A site is an algebraic generalization of a topological space. It consists of a category $\mathscr{C}$ (in analogy to the category $\mathrm{Op}(X)$ of open subsets of a topological space X), endowed with a so-called *Grothendieck topology*, which, for each object $U \in \mathscr{C}$, defines a set $\mathrm{Cov}(U)$ of *coverings* of U (a set of families of morphisms into U). An important example for our purposes is the **subanalytic site** X_{sa} associated to a real analytic manifold X: In contrast to the usual topology, we only consider a certain selection of open subsets and coverings: The underlying category $\mathrm{Op}^c(X_{sa})$ contains the relatively compact, subanalytic open subsets (together with inclusion maps as morphisms) and we admit only finite coverings.

Subanalytic sets have several useful properties to work with special kinds of functions such as tempered functions. They are defined to be locally projections of semianalytic sets:

Definition. *Let X be a real analytic manifold. A subset $Z \subseteq X$ is called semi-analytic if, for any $p \in X$, there is an open neighbourhood $U \subseteq X$ of p such that $Z \cap U = \bigcup_{i \in I} \bigcap_{j \in J} Z_{ij}$ where I and J are finite and $Z_{ij} = \{y \in U | f_{ij} > 0\}$ or $Z_{ij} = \{y \in U \,|\, f_{ij} = 0\}$ for an analytic function f_{ij} on U. We call $Z \subseteq X$ subanalytic if, for any $p \in X$, there exist an open neighbourhood $U \subseteq M$ of p, a real analytic manifold N and a relatively compact semianalytic subset $A \subseteq M \times N$ such that $Z \cap U = \pi(A)$ where $\pi : M \times N \to M$ is the projection.*

Let k be a commutative unital ring. A *presheaf* (of k-modules) on X_{sa} is just a contravariant functor $\mathcal{F}$ from $\mathrm{Op}^c(X_{sa})$ to $\mathrm{Mod}(k)$. It is a sheaf if, for any finite collection $\{U_i\}_{i \in I} \subseteq \mathrm{Op}^c(X_{sa})$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

is exact, where $U := \bigcup_{i \in I} U_i$.

Let $\mathcal{F}$ be a sheaf of k-modules on X. There are different ways to make a sheaf on the subanalytic site out of $\mathcal{F}$:

- The functor ϱ_* (direct image):

$$\varrho_* \mathcal{F}(U) := \mathcal{F}(U) \quad \text{for } U \in \mathrm{Op}^c(X_{sa}) \subseteq \mathrm{Op}(X)$$

- The functor $\varrho_!$:
  $\varrho_! \mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(\overline{U}) := \varprojlim_{\overline{U} \subseteq V \in \mathrm{Op}^c(X_{sa})} \mathcal{F}(V).$$

Tempered functions

We explore a certain class of smooth (complex-valued) functions which do not grow "too fast" near their domain's boundary:

Definition. *Let X be a real analytic manifold and $\mathcal{C}_X^\infty$ its sheaf of complex-valued smooth functions and $U \subseteq X$ open. A function $f \in \mathcal{C}_X^\infty(U)$ is called* tempered *if f and all its derivatives satisfy the following* polynomial growth *condition: For any $p \in X$, there exist a compact neighbourhood $K \subseteq X$ of p and a constant $M \in \mathbb{N}$ such that*

$$\sup_{x \in K \cap U} \mathrm{dist}(x, K \backslash U)^M |f(x)| < \infty.$$

The subset of $\mathcal{C}_X^\infty(U)$ consisting of tempered functions is denoted by $\mathcal{C}_X^{\infty,t}(U)$.

The assignment $U \mapsto \mathcal{C}_X^{\infty,t}(U)$ does not define a sheaf on X because glueing tempered functions does not always result in a function which is again tempered. However, tempered smooth functions form a sheaf on the subanalytic site X_{sa}, which we denote by $\mathcal{C}_{X_{sa}}^{\infty,t}$. This sheaf is a $\varrho_! \mathcal{D}_X$-module.

We define the *sheaf of* **tempered holomorphic functions** to be the complex of $\varrho_! \mathcal{D}_X$-modules

$$\mathcal{O}_{X_{sa}}^t := R\mathcal{H}om_{\varrho_! \mathcal{D}_{\bar{X}}}\left(\varrho_! \mathcal{O}_{\bar{X}}, \mathcal{C}_{X_{sa}}^{\infty,t}\right),$$

which is concentrated in degree 0 if X is a complex curve. Here, $\bar{X}$ denotes the complex conjugate manifold.

Similarly to our previous definition of solution complexes, we call

$$\mathscr{S}ol^t(\mathcal{M}) := R\mathcal{H}om_{\varrho_! \mathcal{D}_X}\left(\varrho_! \mathcal{M}, \mathcal{O}_{X_{sa}}^t\right)$$

the **tempered solutions** of $\mathcal{M}$.

An example of tempered solutions

The example of Kashiwara and Schapira (2003) is very suitable for understanding the importance of tempered solutions and the subanalytic site.

Let $X = \mathbb{C}$ and $P = z^2 \frac{\partial}{\partial z} + 1$. Then by Picard-Lindelöf the holomorphic solutions in degree 0 are

$$\mathscr{S}(\mathcal{M}_P) = \mathbb{C}_{X \backslash \{0\}} \cdot \exp(1/z).$$

To calculate $\mathscr{S}^t(\mathcal{M}_P) := \mathrm{H}^0 \mathscr{S}ol^t(\mathcal{M}_P)$, we prove a statement about the temperedness of $\exp(1/z)$:

Proposition. *Let $U \in \mathrm{Op}^c(X_{sa})$. Then $\exp(1/z)$ is tempered on U if and only if there is an $A > 0$ such that $\mathrm{Re}\left(\frac{1}{z}\right) < A$ on U, i. e. $U \subseteq U_A$ for*

$$U_A := \left\{z \in \mathbb{C} \,\Big|\, \mathrm{Re}\left(\frac{1}{z}\right) < A\right\}.$$

A consequence is the following formula:

$$\mathscr{S}^t(\mathcal{M}_P) \cong \varinjlim_{A > 0} \varrho_* \mathbb{C}_{U_A}$$

It is remarkable that this limit does not commute with the functor ϱ_* and hence $\mathscr{S}^t(\mathcal{M}_P) \not\cong \varrho_* \mathbb{C}_{X \backslash \{0\}}$.

This example also shows how tempered solutions *refine* the notion of holomorphic solutions:
If we consider the operator $Q = z\left(z\frac{\partial}{\partial z} + 1\right)$, one can prove that the holomorphic solutions of $\mathcal{M}_P$ and $\mathcal{M}_Q$ are isomorphic, but $\mathscr{S}^t(\mathcal{M}_Q) \cong \varrho_* \mathscr{S}(\mathcal{M}_Q) \cong \varrho_* \mathbb{C}_{X \backslash \{0\}}$, so we can indeed distinguish P and Q by their tempered solutions.

References

[1] A. HOHL: Sheaves on the subanalytic site and tempered solutions of $\mathcal{D}$-modules on curves. Bachelorarbeit, Universität Augsburg, September 2014.

[2] R. HOTTA, K. TAKEUCHI and T. TANISAKI: D-Modules, Perverse Sheaves and Representation Theory. Progress in Math. 236. Boston: Birkhäuser, 2008.

[3] M. KASHIWARA and P. SCHAPIRA: Ind-sheaves. In: Astérisque 271 (2001).

[4] M. KASHIWARA and P. SCHAPIRA: Microlocal study of ind-sheaves I: micro-support and regularity. In: Asterisque 284 (2003).