Simulation of a Phase-Field Model in Fracture Mechanics

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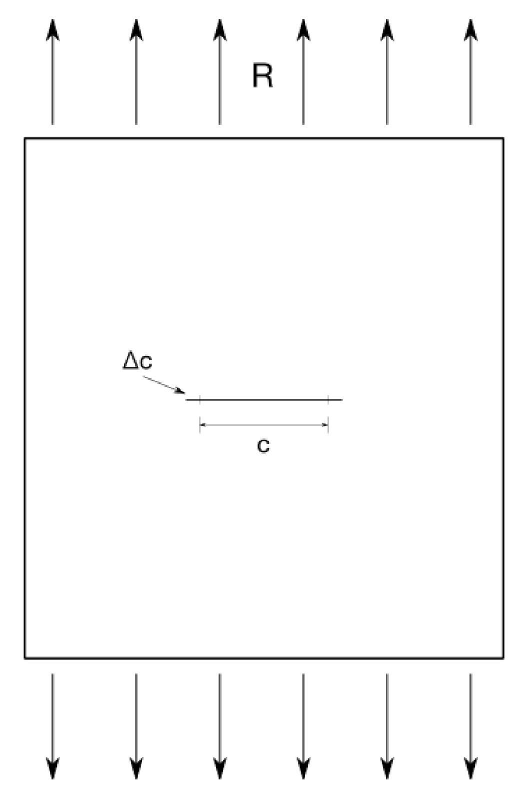
Abstract

The prediction of critical loads is an important part of the design process of any car-component, bridge, etc. However, such loads cannot be predicted by the theory of elasticity alone. Further assumptions about the interaction between elastic energy and surface energy is necessary. Although such models have been available since the 1920s, [1], they always required an a-priori known crack-path. In 1998, Francfort and Marigo [2] presented an extension of Griffith's original model, which no longer required such assumptions. They do so by minimizing a functional E which unfortunately is not easily accessible to numerical approximation. So E is approximated by a family of functionals $(E_\varepsilon)_\varepsilon$, which can be minimized with standard finite element techniques. Some comparisons between numerical and experimental results [3] are presented below.

Griffith's criterion (1921)



An infinitely large plate is loaded with a Force R at its ends. The plate contains an initial crack of length c. The initial crack-length is increased by an infinitesimal amount Δc. Griffith's criterion then states that the crack will grow iff $\frac{\Delta E}{\Delta c} < 0$, where E is the total energy and

$$E := \underbrace{E_b}_{elastic energy} + \underbrace{kc}_{surface energy}.$$

Here k is some arbitrary, strictly positive constant, also called *fracture toughness*.

The Model by Francfort and Marigo (1998)

Because the above model requires an a priori known crack-path, an extension is required and was presented by Francfort and Marigo in 1998.

$$\min_{u,\Gamma} \int_{\Omega\setminus\Gamma} \frac{1}{2}\mu|\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(\Gamma)$$
$$u = g(t) \text{ auf } \partial_d\Omega$$
$$\partial_n u = 0 \text{ auf } \partial\Omega \setminus \partial_d\Omega$$

where $u \in H^1(\Omega)$ and $\Gamma \subset \Omega$, Γ closed. This model can handle unknown crack-paths but it is not clear how to deal with arbitrary sets Γ numerically.

Approximation via Γ-convergence

Definition 1 *Suppose that E and $(E_\varepsilon)_\varepsilon$ are functionals acting on a function space X, which is assumed to be equipped with a topology τ. Then E_ε is said to $\Gamma(\tau)$-converge to E for $\varepsilon \to 0$ iff every $u \in X$ fulfils the following properties:*

1. Every sequence $(u_\varepsilon)_\varepsilon \subset X$ that has a limit point u w.r.t. τ satisfies:

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E(u).$$

2. For every u there exists a sequence $(u_\varepsilon)_\varepsilon \to u$ (w.r.t. τ) s.t.

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \leq E(u).$$

Theorem 1 *Let $(E_\varepsilon)_\varepsilon$, E and X be defined as above and $(u_\varepsilon)_\varepsilon$ be a sequence of minimizers of the E_ε, $\varepsilon \to 0$. Furthermore, let either $(u_\varepsilon)_\varepsilon$ or X be compact. Assume that $E_\varepsilon \xrightarrow{\Gamma} E$. Then (up to a subsequence), $(u_\varepsilon)_\varepsilon$ converge to a minimizer u of E and $\lim E_\varepsilon(u_\varepsilon) = E(u)$*

Instead of minimizing the "awkward" functional from above, the following functional is minimized. It can then be proven that E_ε Γ-converges to a slightly modified version of E for $\varepsilon \to 0$.

$$\min_{u,v} E_\varepsilon(u,v) := \int_\Omega \frac{1}{2}\mu v^2|\nabla u|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{4\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2 \,\mathrm{d}x$$

where $u, v \in H^1(\Omega)$.

Interpretation of the auxiliary Variable v

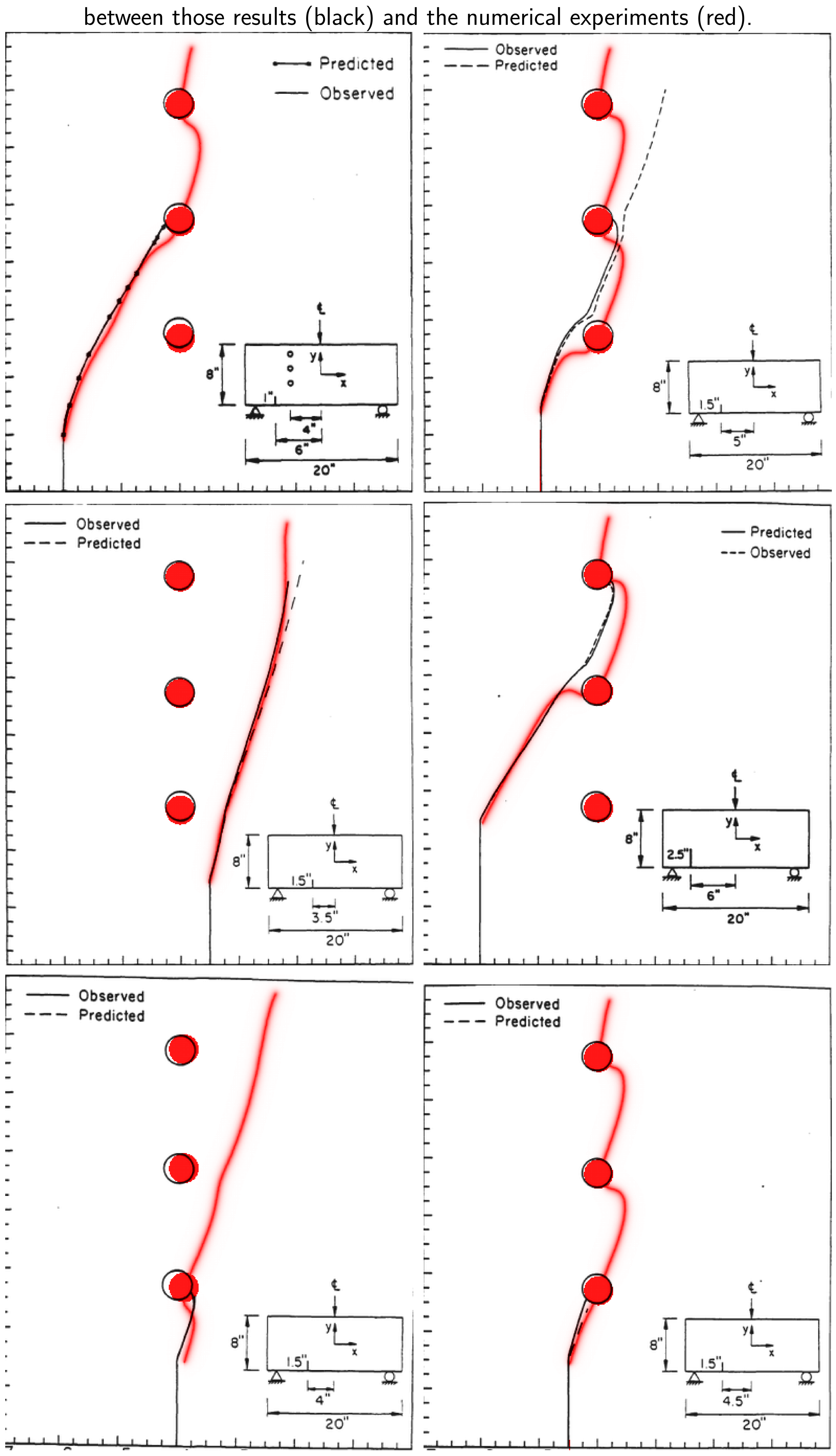
The variable v acts as a damage variable with $v = 0$ representing completely damaged material and $v = 1$ representing undamaged material. The solution of the above minimization problem for the 1D case can be computed analytically and is (up to a few constants)

$$v_{min}(x) = 1 - e^{-\frac{|x|}{\varepsilon}}$$

So for $\varepsilon \to 0$ v_{min} converges to a function which is 1 everywhere exept for $x = 0$, which is the "crack-set".

Some results

In [3], some experimental results were carried out. The following plots show a comparison between those results (black) and the numerical experiments (red).



References

1. Griffith, A. A.: The phenomena of rupture and flow in solids, Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character $221 : 163 - 198$ (1921).
2. Francfort, G. A.; Marigo, J.-J.: Revisiting brittle fracture as an energy minimization problem. Journal fo the Mechanics and Physics of Solids $46 : 1319 - 1342$ (1998).
3. Wu, S.; Ingraeffa, A. R.; Grigoriu, M.: Probabilistic fracture mechanics - a validation of predictive capability. School of Civil and Environmental Engineering; Cornell University; August 1990

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