Lévy-driven tempo-spatial Ornstein-Uhlenbeck processes

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Abstract

We extend the Lévy-driven Ornstein-Uhlenbeck process as a timewise process to time and space. This is achieved by employing stochastic Volterra integral equations in time and space, which comprise a stochastic integral with respect to a Lévy basis. We formulate conditions for the existence and uniqueness of the solution and derive an explicit solution formula. After giving criteria for stationarity of these processes, we establish the second order structure in the stationary case by means of this solution formula. The theoretical results are illustrated by concrete examples. For further details we refer to [3].

Motivation

A Lévy-driven Ornstein-Uhlenbeck (OU) process is defined as the unique solution of the stochastic integral equation

$$X(t) = \int_0^t -\lambda X(s)\,\mathrm{d}s + \int_0^t \mathrm{d}L(s), \quad t \geq 0,$$

where $\lambda > 0$ and L is a Lévy process, i.e. L has independent and stationary increments and càdlàg paths (see e.g. [1, Section 4.3]).

Goal: Generalization to a stochastic process in time and space.

Idea: We generalize the defining stochastic integral equation to:

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s,\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y), \qquad (1)$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, μ is a measure on $\mathbb{R}^+ \times \mathbb{R}^d$, $g\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is deterministic, and Λ is a Lévy basis.

The deterministic convolution Volterra integral equation

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s,x-y)\mu(\mathrm{d}s,\mathrm{d}y) + f(t,x) \qquad (2)$$

The **forcing function** f does not depend on X (see [2, Section 4.1]).

Lemma: Let $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\mu(\{0\} \times \mathbb{R}^d) = 0$. Then there exists a unique measure $\rho \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, called **the resolvent**, such that

$$\rho + \mu = \mu * \rho.$$

Theorem: Let $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ with $\mu(\{0\} \times \mathbb{R}^d) = 0$. Then for every $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ the unique solution in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ to (2) is

$$X = f - f * \rho.$$

Stochastic integration w.r.t. Lévy bases

Definition: A stochastic process $(\Lambda(B))_{B \in \mathcal{B}_b}$ is called a **homogeneous Lévy basis** if

1) for disjoint $(B_i)_{i\in\mathbb{N}}$ in $\mathcal{B}_b$ satisfying $\bigcup_{i=1}^\infty B_i \in \mathcal{B}_b$ we have

$$\Lambda\Big(\bigcup_{i=1}^\infty B_i\Big) = \sum_{i=1}^\infty \Lambda(B_i) \quad \text{a.s.}$$

2) $(\Lambda(B_i))_{i\in\mathbb{N}}$ are independent for disjoint $(B_i)_{i\in\mathbb{N}}$ in $\mathcal{B}_b$

3) $\Lambda(B) \overset{\mathrm{d}}{=} \Lambda(\tilde{B})$ for all $B, \tilde{B}$ in $\mathcal{B}_b$ such that $\mathrm{Leb}(B) = \mathrm{Leb}(\tilde{B})$

By the Lévy-Khintchine formula we get for the characteristic function:

$$\Phi(\Lambda(B))(u) = \exp\left\{\mathrm{Leb}(B)\Big[iub - \frac{1}{2}u^2C + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz\mathbb{1}_{(-1,1)}(z))\nu(\mathrm{d}z)\Big]\right\},$$

where $b \in \mathbb{R}$, $C \in \mathbb{R}^+$ and ν is a **Lévy measure** on $\mathbb{R}$.

Then (b, C, ν) is called **the characteristic triplet**.

Definition: A measurable function $h : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is called Λ-**integrable** if there exists a sequence of simple functions (h_n) such that

1) h_n converges to h Leb-a.e.,

2) $(\int_B h_n \,\mathrm{d}\Lambda)$ converges in probability for all $B \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d)$.

In this case we define:

$$\int_B h \,\mathrm{d}\Lambda = \text{P-}\lim_{n\to\infty} \int_B h_n \,\mathrm{d}\Lambda.$$

This definition does not specify the class of Λ-integrable functions. However, there are convenient **integrability conditions** (see [4, Section 2]).

The stochastic convolution Volterra integral equation

Goal: Solve equation (1).

Idea: Imitate the deterministic theory pathwise.

Theorem: Let

- $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ with $\mu(\{0\} \times \mathbb{R}^d) = 0$,
- $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ be bounded, and
- Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments.

Then the unique solution (up to versions) to (1) is given by

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(t-s, x-y)\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

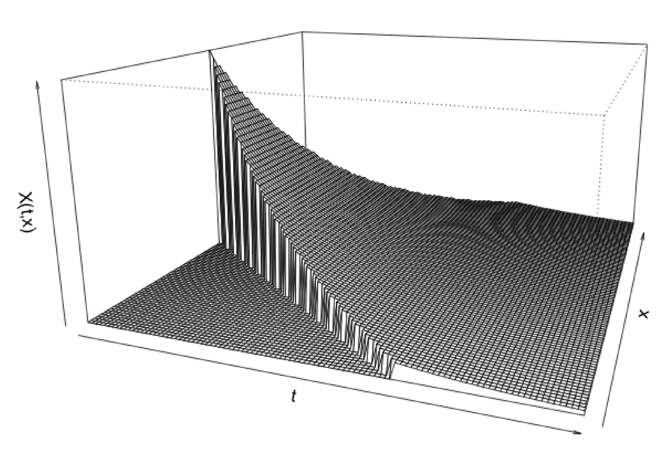
Example

$$X(t,x) = \int_0^t -\lambda X(s,x)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y) e^{\frac{-\lambda\|x-y\|}{c}} \Lambda(\mathrm{d}s,\mathrm{d}y) \qquad (3)$$

$$A(t,x) = A + (t,x) \qquad A = \{(t,x) \in \mathbb{R}^- \times \mathbb{R}^d : \|x\| \leq c|t|\}$$

Theorem: The unique solution to equation (3) is given by

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y) e^{-\lambda(t-s)} \Lambda(\mathrm{d}s,\mathrm{d}y). \qquad (4)$$



A sample of the tempo-spatial evolution of a single Lévy jump. The peak belongs to the point of occurrence of the Lévy jump in space-time. An exponential decay in time and a uniform propagation in space are observable. For pure-jump Lévy bases X in equation (4) can be understood as the superposition of a large number of these jump effects.

Stationarity

Problem: No stationary solutions so far.

Idea: Modify the stochastic integral equation to:

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s,x-y)\mu(\mathrm{d}s,\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) + V(t,x) \quad (5)$$

Theorem: Under the additional assumption that $g * (\delta_0 - \rho) \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ is bounded there exists a stochastic process V on $\mathbb{R}^+ \times \mathbb{R}^d$ such that equation (5) has a unique (up to versions) strictly stationary solution, namely

$$X(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

Second order structure

Theorem: The second order structure of the strictly stationary solution X is given by

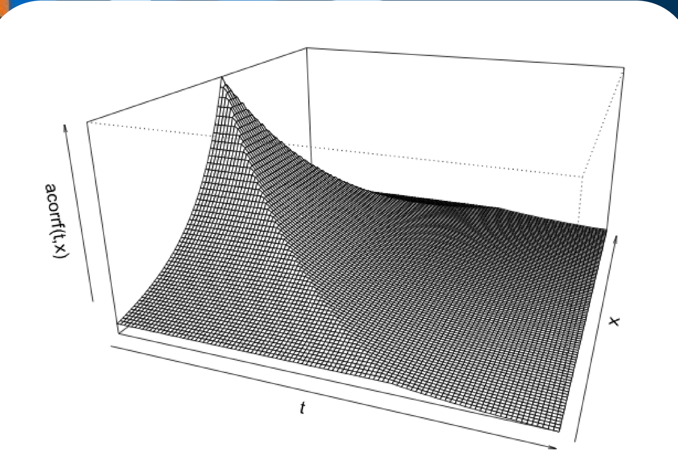
$$\mathbb{E}(X(0,0)) = \kappa_1 \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(s,y)\,\mathrm{d}s\mathrm{d}y,$$

$$\mathrm{acf}(\tilde{t},\tilde{x}) = \mathrm{Cov}(X(t,x), X(t+\tilde{t}, x+\tilde{x}))$$

$$= \kappa_2 \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(s,y)(g * (\delta_0 - \rho))(s+\tilde{t}, y+\tilde{x})\,\mathrm{d}s\mathrm{d}y,$$

for all $t, \tilde{t} \in \mathbb{R}^+$, $x, \tilde{x} \in \mathbb{R}^d$,

where $\kappa_1 = b + \int_{\mathbb{R}\setminus(-1,1)} x\,\nu(\mathrm{d}x) \in \mathbb{R}$ and $\kappa_2 = C + \int_{\mathbb{R}} x^2\,\nu(\mathrm{d}x) \in \mathbb{R}^+$.



The autocorrelation function of the strictly stationary solution in the example: $\mathrm{acorrf}(t,x) = \min(\exp(-\lambda t), \exp(-\frac{\lambda|x|}{c}))$

Notation

$M(\mathbb{R}^+ \times \mathbb{R}^d)$	signed complete Borel measures on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite total variation
$M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$	signed measures on $\mathbb{R}^+ \times \mathbb{R}^d$ lying in $M([0,T] \times \mathbb{R}^d)$ when restricted to $[0,T] \times \mathbb{R}^d$ for all $T \geq 0$
$\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$	functions $f\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ s.t. $f \in \mathrm{L}^1([0,T] \times \mathbb{R}^d)$ for all $T \geq 0$
$\mathcal{B}_b$	bounded Borel sets in $\mathbb{R}^+ \times \mathbb{R}^d$

References

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[4] B.S. Rajput and J. Rosiński. Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields*, 82(3): 451–487, 1989.

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