Open Books and Contact Structures

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Abstract

In this paper we worked on open books supporting contact structures and stable Hamiltonian structures in dimension three. Every stable Hamiltonian structure is stable homotopic to one that is supported by an open book. So if every open book with given signs on the binding would support a positive contact structure, the homotopy classification of the stable Hamiltonian structures could be reduced to the classification of positive contact structures. This is in fact not the case but to get a better understanding of both classifications we will study what kind of decorated open books support a contact structure with given signs on the binding. In the end we will find a way to compute the fundamental group of open books in general.

Open Books

There are two different, but in the end equivalent, ways to define open books. Depending on the situation it is helpful to look at a certain one so they both have their reason of existence. From now on M is a closed and oriented 3-manifold.

Definition. (Open Book Decomposition)
An abstract open book is a pair (B, π) with the following properties:

1. B is an oriented link in M and
2. π is a fibration of $M \setminus B$ over S^1 such that, for all $\theta \in S^1$, $\pi^{-1}(\theta)$ is the interior of a 2-dimensional submanifold Σ_θ with boundary $\partial\Sigma_\theta = B$. Σ_θ is called a *page* and B is called the *binding* of the book.

Definition. (Abstract Open Books)
An abstract open book is a pair (Σ, ϕ) with the following properties:

1. Σ is a compact oriented 2-dimensional submanifold with boundary and
2. $\phi\colon \Sigma \to \Sigma$ is a diffeomorphism with $\phi = id_\Sigma$ in a neighborhood of $\partial\Sigma$ and is called the *monodromy*.

With this pair and a diffeomorphismus $\psi\colon S^1 \times \partial D^2 \to Z \times S^1$, where Z is a boundary component of Σ, we can construct a closed and oriented 3-manifold M_ϕ:

$$M_\phi = \Sigma_\phi \cup_\psi \left(\coprod_{|\partial\Sigma|} S^1 \times D^2 \right)$$

Here Σ_ϕ denotes the mapping torus induced by the map ϕ, which is the quotient space

$$\Sigma \times [0,1] \big/ (\phi(x), 0) \sim (x, 1)$$

and $|\partial\Sigma|$ denotes the number of boundary components of Σ.
The glueing diffeomorphism ψ is by definition the unique diffeomorphism (up to isotopy) that maps $S^1 \times \{p\}$ onto $Z \times \{y\}$ (this is well-defined since $\phi = id_\Sigma$ on the $\partial\Sigma$) where $p \in \partial D^2$ and $y \in [0,1]$ and $\{q\} \times \partial D^2$ onto $\{z\} \times {}^{[0,1]}\!/\!\sim$ where $q \in S^1$ and $\{z\} \in Z$.

Contact Structures

Definition. (Plane Field)
A plane field ξ on a 3-manifold M is a subbundle of the tangent bundle TM where, for all $p \in M$, $\xi_p (= T_pM \cap \xi)$ is a 2-dimensional subspace.

Lemma. *If M and ξ are oriented there exists a 1-form that is defined on M and $\xi_p = \ker\alpha$ for all $p \in M$.*

Definition. (Contact Structure)
If for any 1-form α that represents the plane field ξ the following holds true

$$\alpha \wedge d\alpha \neq 0$$

then ξ is called a *contact structure*. In this case α is called a *contact form* of the contact structure ξ. If $\alpha \wedge d\alpha > 0$ then ξ is called a *positive contact structure*.

Connection between the Two

At first glance these two concepts don't seem to bo affiliated with each other. But the following definition will build a meaningful connection as we will see.

Definition. (Open Book supporting a Contact Structure)
An open book (B, π) of manifold M is said to *support* a contact structure ξ on M if ξ is isotopic to a contact structure ξ_1 with a corresponding 1-form α_1 such that

1. $\alpha_1 > 0$ on the binding B of the open book and
2. $d\alpha_1$ is a positive volume form on every page Σ_θ of the open book.

The definition of *open books supporting contact structures* points up a strong connection between these two concepts in dimension three, as Thurston and Winkelnkemper (1) and Giroux (2-4) could show:

1. Every open book supports a contact structure
2. Two contact structures supported by the same open book are isotopic
3. Every contact structure is supported by an open book

We concentrate on the first connection.

The questions we ask ourselves

Now these are the basic concepts. We are interested in the implications we get by solely requiring the second property of the Definition above ($d\alpha$ is a positive volume form on Σ). E.g. is it possible to construct a positive contact structure that satisfies the second property but not the first? Does there exist a contact structure that is zero or negative on some or even all binding components of the open book? And if there are some. what would the resulting 3-manifold look like?

Some Answers

K. Cieliebak and E Volkov [1] found out that there exists no contact structure that has negative sign on all binding components, and furthermore have proven the following theorem.

Theorem. *Let (Σ, Φ) be a decorated open book for a 3-manifold M. The monodromy associated to its abstract open book is a suitable isotopy of Φ using Dehn twists. Now assume that we have a 1-form λ on Σ and there exist $\varepsilon, \delta > 0$ with the following properties:*

- *i) λ restricts as $-(\delta + s)d\zeta$ near the negative boundary components in $\bigcup_{j=1,\dots,k}\{(\zeta, s) \in S^1 \times [0,1] \,:\, s \leq R_j\}$ and*
- *ii) as $(1+s)d\zeta$ near the positive boundary components,*
- *iii) $\frac{\Phi^*\lambda\wedge\lambda}{d\lambda} > -\varepsilon$,*
- *iv) $\varepsilon < \delta + \min\limits_{j=1,..,k} R_j$,*
- *v) $d\lambda > 0$ on the pages of the open book and*
- *vi) Φ_1 preserves the volume form $d\lambda$.*

Then there exists a positive contact form α on M such that $(d\alpha, \alpha)$ is supported by the open book (Σ, Φ).

It can be shown that for a class of open books that this is actually the case. A simple case is the open book $([a,b] \times S^1, D_c^{-1})$. Here D_c^{-1} is the negative Dehn twist along the meridian c of the anulus $[a,b] \times S^1$. Due to the following theorem we can compute the fundamental group of the resulting 3-manifold. It is the trivial group and hence the 3-manifold is homeomorphic to the S^3.

Theorem. *The fundamental group of the open book (Σ, ϕ) is given by*

$$\begin{aligned} \pi_1(M_\phi) = \langle b_1, \dots, b_k | r_1, \dots, r_m, \\ b_i\phi(b_i)^{-1} \quad &\text{for i=1,...,k ,} \\ \phi(\alpha_j)\alpha_j^{-1} \quad &\text{for j=2,...,n}\rangle \end{aligned}$$

where

$$\pi_1(\Sigma) = \langle b_1, \dots, b_k | r_1, \dots, r_m \rangle$$

and n is the amount of boundary components $\partial_j\Sigma$. The base point p is chosen in a neighborhood of an arbitrary fixed first boundary component $\partial_1\Sigma$ where ϕ is the identity. The α_j are then paths from said base point p to a point p_j that lies in the neighborhood of boundary component $\partial_j\Sigma$ where ϕ is also the identity.

Outlook

It would be interesting to find more implications on the contact form α. That could either be done by finding a theorem with less or different requirements or finding further classes of open books that fulfill the requirements of the theorem of Cieliebak and Volkov. This area of research does not seem to be exhausted.

References

[1] K. Cieliebak and E. Volkov, *Stable Hamitlonian structures in dimension three are supported by open books*, arXiv:1012.3854
[2] J. Etnyre, *Lectures on open book decompositions and contact structures*, arXiv:math/0409402
[3] W. Thurston and H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52**, 345-347 (1975).

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