Sampling q-colorings on graphs: convergence and cutoff

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Proper q-colorings

Given a graph $G = (V, E)$ and a finite set of spins Σ, a **spin system** π is a probability distribution on Σ^V. The special case of

$$\pi(\sigma) := \frac{1}{Z} \exp\left(\sum_{\{u,v\}\in E} g_{u,v}(\sigma(u), \sigma(v)) \right), \tag{1}$$

with $\Sigma = \{1, \ldots, q\}$, normalizing constant Z and **nearest neighbor interactions**

$$g_{u,v}(\sigma(u), \sigma(v)) = \begin{cases} 0 & \text{if } \sigma(u) \neq \sigma(v) \\ -\infty & \text{else} \end{cases} \tag{2}$$

is called the **proper q-coloring model**. All elements $\sigma \in \Sigma^V$ with $\pi(\sigma) > 0$ are referred to as **proper q-colorings**. Intuitively, this are all assignments of colors to the vertices such that every two vertices which share an edge do not have the same color.

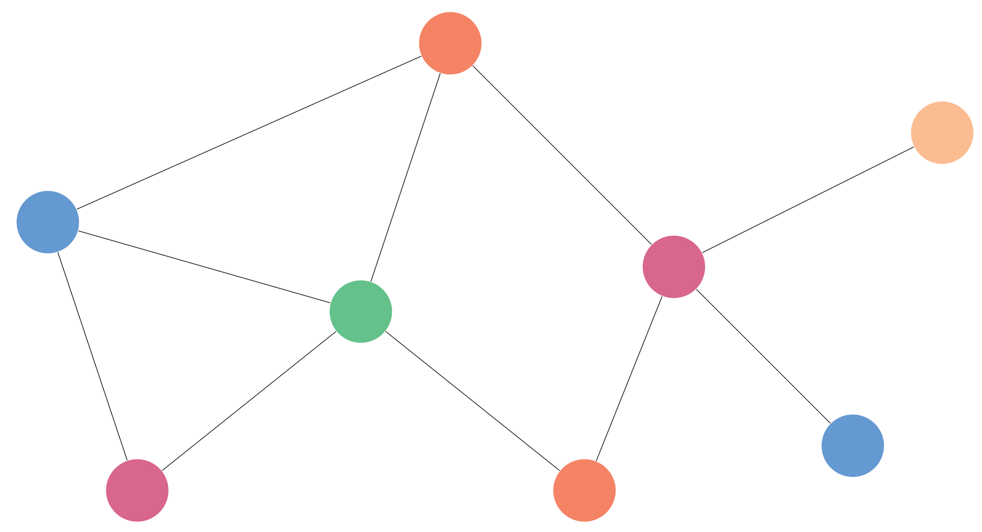

Fig. 1: Proper q-coloring using $q = 5$ colors

Our goal is to provide a sample according to the proper q-coloring model, i.e. a proper q-coloring chosen uniformly at random among all possible proper q-colorings.

Sampling from the proper q-coloring model

Given an initial proper q-coloring, define a Markov chain according to one of the following transition mechanisms:

Glauber dynamics for proper q-colorings

1. Choose $v \in V$ uniformly at random
2. Choose color of v uniformly among all colors not taken by some neighbor of v

Metropolis sampler for proper q-colorings

1. Choose $v \in V$ and $c \in \Sigma$ independently and uniformly at random
2. Set color of v to c if this gives a proper q-coloring and keep the current configuration else

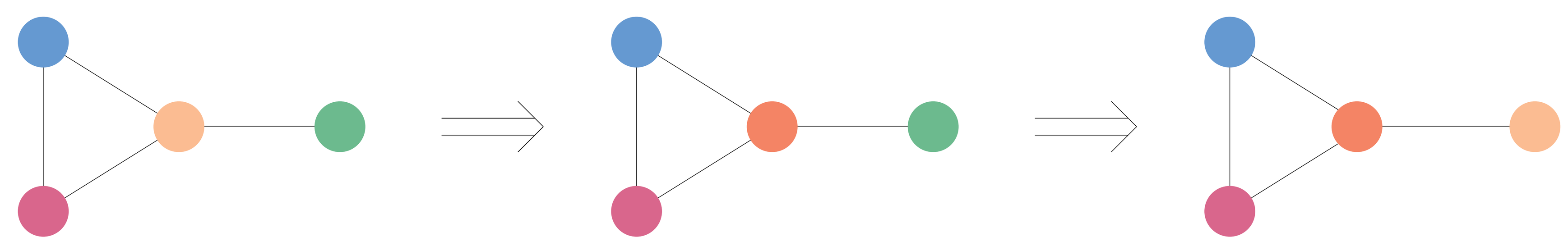

Fig. 2: Two possible steps of Glauber dynamics with $q = 5$ colors.

Theorem. *For any finite graph $G = (V, E)$ with maximum degree Δ and $q > \Delta + 1$ colors, both samplers converge towards the uniform distribution over all proper q-colorings on Σ^V.*

Speed of convergence

For a Markov chain (X_t) on a state space Ω with transition matrix P and stationary distribution π, we define the **mixing time** $t_{\text{mix}}(\varepsilon) := \max_{x\in\Omega} \|P_x(X_t \in \cdot) - \pi(\cdot)\|_{\text{TV}}$ as a quantitative measure of how fast the Markov chain actually converges to π.

Theorem (Bubley/Dyer, 1997). *Consider the proper q-coloring model on $G = (V, E)$ with maximum degree Δ for $q > 2\Delta$ colors. Then the mixing time of Glauber dynamics satisfies*

$$t_{\text{mix}}(\varepsilon) \leq \left\lceil \left(\frac{q - \Delta}{q - 2\Delta} \right) n \log\left(\frac{n}{\varepsilon}\right) \right\rceil, \tag{3}$$

where n denotes the number of vertices in G.

Idea of the proof: Analyze a coupling for configurations which differ in precisely one vertex.

Perfect sampling

Goal: Provide a sample *exactly* according to the proper q-coloring model.
For Glauber dynamics (X_t^x) starting in $x \in \Sigma^V$, we define

Grand coupling: Coupling of (X_t^x) for all initial $x \in \Sigma^V$ simultaneously

Disagreement process: $\mathcal{H}_t(u) := \begin{cases} 1 & \text{if } \exists x, x' : X_t^x(u) \neq X_t^{x'}(u) \\ 0 & \text{else} \end{cases}$

Complete coupling: At t^* it holds that $\mathcal{H}_{t^*}(u) = 0$ for all $u \in V$

Proposition. *The configuration drawn at a (random) time t^*, where we have detected complete coupling, is a sample exactly according to the stationary distribution.*

Bounding chain approach

How can we detect complete coupling for the proper q-color model efficiently?
Idea: Define the **bounding chain** to be a Markov chain (Y_t) on $(2^\Sigma)^V$ such that for any (X_t^x) in the grand coupling

$$X_t^x(v) \in Y_t(v) \tag{4}$$

holds for all $t \geq 0$ and $v \in V$. We have complete coupling at t^* if $\prod_{v\in V} |Y_{t^*}(v)| = 1$.

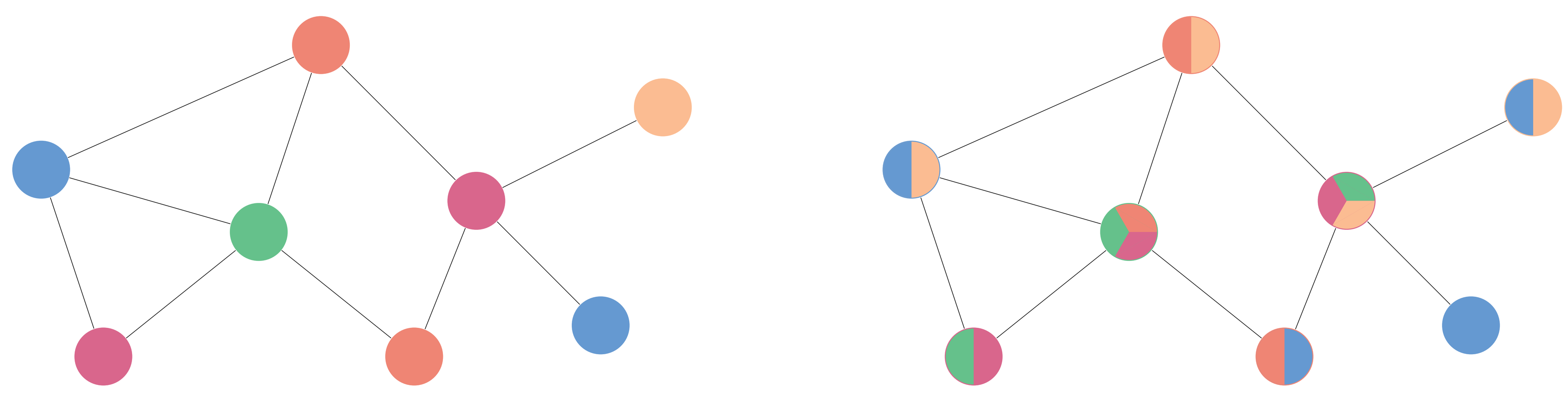

Fig. 3: Proper q-coloring with possible bounding chain

Theorem (Huber, 1999). *Consider the proper q-coloring model on $G = (V, E)$ with maximum degree Δ and $|V| = n$ for $q \geq \Delta(\Delta + 2)$ colors. Then for $t \geq \frac{1}{\beta}(k + \log n)n$ with $0 < \beta < 1$ constant and $k \in \mathbb{N}$ arbitrary, it holds that*

$$\mathbb{P}(\textit{complete coupling not detected until time } t) \leq e^{-k} \tag{5}$$

Cutoff phenomenon

A sequence $(X_t^n)_{n\in\mathbb{N}}$ of irreducible and aperiodic Markov chains exhibits **cutoff** if for all $0 < \varepsilon < 1$

$$\lim_{n\to\infty} \frac{t_{\text{mix}}^{(n)}(\varepsilon)}{t_{\text{mix}}^{(n)}(1 - \varepsilon)} = 1 \tag{6}$$

A sequence $(w_n)_{n\in\mathbb{N}}$ is a **cutoff window** for $(X_t^n)_{n\in\mathbb{N}}$ if $w_n = o(t_{\text{mix}}^{(n)})$ and

$$t_{\text{mix}}^{(n)}(\varepsilon) = t_{\text{mix}}^{(n)}(1 - \varepsilon) + \mathcal{O}(w_n) \tag{7}$$

holds for all $0 < \varepsilon < 1$.

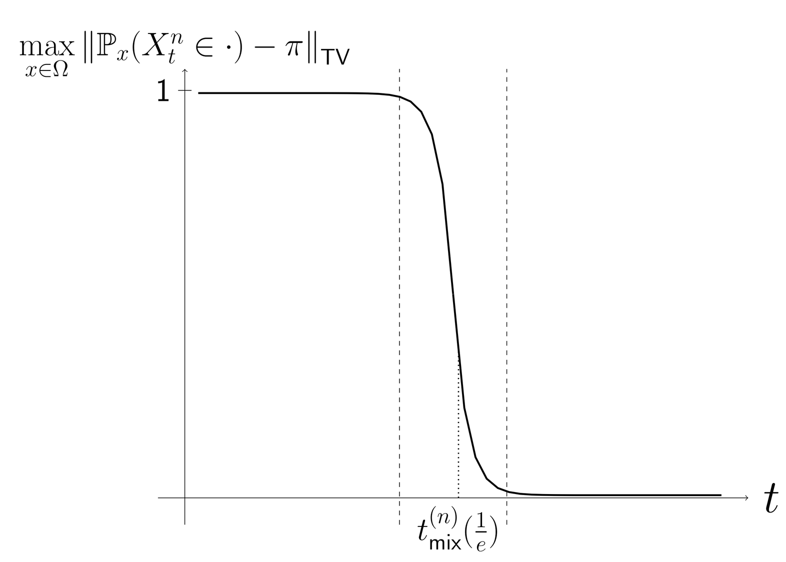

Fig. 4: Cutoff behavior for a Markov chain (X_t^n) with n large

Cutoff phenomenon for Glauber dynamics

Theorem (Lubetzky/Sly, 2012). *Consider (continuous-time) Glauber dynamics for the proper q-coloring model on a sequence of boxes $\Lambda^n \subset \mathbb{Z}^d$ of side length n. Suppose that the corresponding disagreement process $\mathcal{H}_t$ satisfies*

$$\max_{u\in\Lambda^n} \mathbb{P}(\mathcal{H}_t(u) = 1) \leq ce^{-Ct} \tag{8}$$

for all $t \geq 0, n \in \mathbb{N}$ and constants $c, C > 0$, then the dynamics exhibits cutoff with a window of $\mathcal{O}(\log\log n)$. In particular, (8) *holds for $q \geq \Delta(\Delta + 2)$ colors.*

Key ideas for the proof:

1) Break dependencies between the vertices

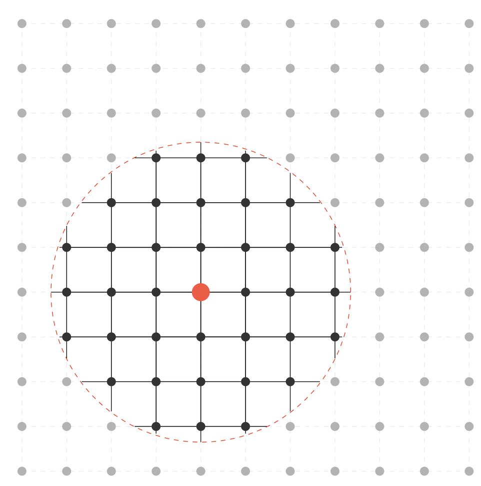

Fig. 5: Consider only color changes within a ball of radius $\mathcal{O}(\log n)$

2) Analyze the mixing behavior on sparse sets

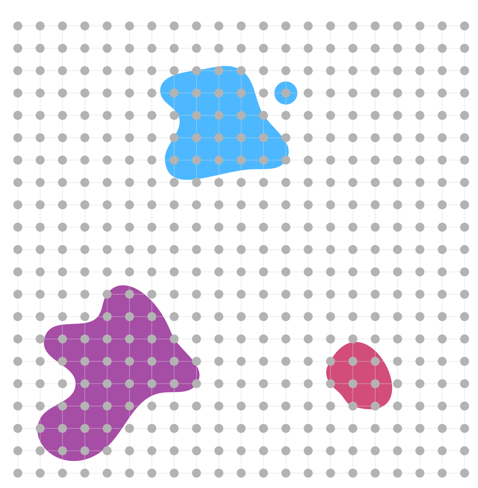

Fig. 6: Sparse set consisting of well separated components

3) Reduce L^1-distances to L^2-distances

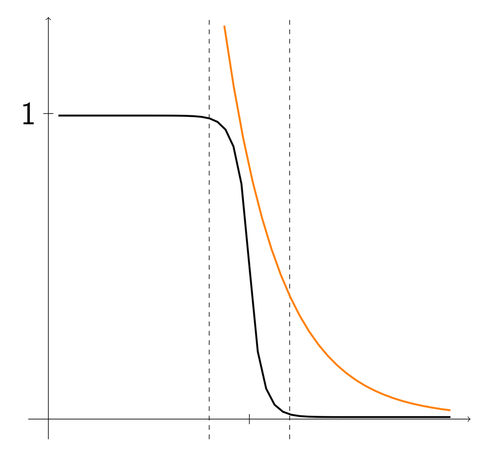

Fig. 7: Behavior of the different distances for large n if cutoff occurs

Cutoff for Metropolis sampler

Corollary. *For $q \geq \Delta(\Delta+2)$ colors, the Metropolis sampler for the proper q-coloring model on a sequence of boxes $\Lambda^n \subset \mathbb{Z}^d$ of length n exhibits cutoff with a window of $\mathcal{O}(\log\log n)$.*

Idea of the proof: Use that the Metropolis sampler can be seen as a time-shifted version of Glauber dynamics in the special case of proper q-colorings.

References

1. D. Levin, Y. Peres, and E. Wilmer. *Markov chains and mixing times.* Provi dence, R.I. American Mathematical Society, 2009.
2. E. Lubetzky and A. Sly. *Cutoff for general spin systems with arbitrary boundary conditions.* Communications on Pure and Applied Mathematics, 2014.