Deformations and Resolutions of Rational Double Points

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Abstract

Let k be an algebraically closed field of characteristic $p \geq 0$. Both canonical surface singularities and certain simple algebraic groups are classified by simply laced Dynkin diagrams. Quite surprisingly, under some restrictions on p, it turns out that there is a more fundamental relation between these two kinds of objects, which was discovered by Brieskorn [2] ($p = 0$), Slodowy [5] ($p \gg 0$ very good) and Shepherd-Barron [3] (p very good). More precisely, the nilpotent variety of the Lie algebra of a simple algebraic group restricted to a transverse slice to the orbit of a subregular nilpotent element under the adjoint action is a rational double point of the corresponding Dynkin type. In turn, this relation can be used to study deformations of rational double points and their simultaneous resolutions using Grothendieck's resolution of the adjoint quotient.

Rational double points

Definition/Theorem. *A normal surface singularity (X,x) is called a* rational double point *if it satisfies one of the following equivalent conditions.*

(i) (X,x) *is a canonical surface singularity.*

(ii) (X,x) *is rational of multiplicity $\mu_x(X) = 2$.*

(iii) (X,x) *is rational of embedding dimension $\mathrm{ed}_x(X) = 3$.*

(iv) (X,x) *is rational and Gorenstein.*

(v) (X,x) *is of multiplicity $\mu_x(X) = 2$, of embedding dimension $\mathrm{ed}_x(X) = 3$ and it can be resolved by successive blow-ups of points.*

(vi) *The completion $\widehat{\mathscr{O}}_{X,x}$ is isomorphic to $k[\![x,y,z]\!]/f_\Gamma$, where f_Γ is one of the equations in the table below if $p \neq 2,3,5$ (see [1] for the equations if $p \in \{2,3,5\}$).*

(vii) *The dual graph of the minimal resolution of (X,x) is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8. Moreover, all irreducible components of the exceptional locus are isomorphic to $\mathbb{P}^1$ and have self-intersection -2.*

Simple algebraic groups

Definition. *A linear algebraic group G is called* simple *if G has no closed connected normal subgroups other than G itself and the trivial subgroup.*

Notation. *In the following, we denote by G a simple algebraic group, by B a Borel subgroup of G, and by $T \subseteq B$ a maximal torus of G with Weyl group W. Moreover, we let $\mathfrak{g}, \mathfrak{b}$ and $\mathfrak{t}$ be the corresponding Lie algebras.*

Using the action of W on the characters of T, one obtains an indecomposable root system Γ and we call Γ *simply laced* if all its roots have the same length. Conversely, any indecomposable root system arises in this way (Chevalley). Simple algebraic groups giving rise to the same root system Γ are related by isogenies and there are unique "biggest" and "smallest" such groups, the *simply connected* resp. *adjoint* simple group of type Γ. Thus, up to isogeny, simple algebraic groups are classified by indecomposable root systems.

Theorem. *Any simply laced and indecomposable root system is of one of the following types: A_n, D_n, E_6, E_7 and E_8 with corresponding Dynkin diagrams.*

Classification

Rational double points | **Simple algebraic groups**

Equation f_Γ for $p \neq 2,3,5$	Resolution graph	Type Γ	Dynkin diagram for Γ	Very good primes for Γ	Simply connected and adjoint simple algebraic group of type Γ
$z^{n+1} + xy$		A_n $n \geq 1$		$p \nmid (n+1)$	$\mathrm{SL}_{n+1} \xrightarrow{(n+1:1)} \mathrm{PGL}_{n+1}$
$z^2 + x^2 y + y^{n-1}$		D_n $n \geq 4$		$p \neq 2$	$\mathrm{Spin}_{2n} \xrightarrow{(4:1)} \mathrm{PSO}_{2n}$
$z^2 + x^3 + y^4$		E_6		$p \neq 2,3$	$(E_6)_{\mathrm{sc}} \xrightarrow{(3:1)} (E_6)_{\mathrm{ad}}$
$z^2 + x^3 + xy^3$		E_7		$p \neq 2,3$	$(E_7)_{\mathrm{sc}} \xrightarrow{(2:1)} (E_7)_{\mathrm{ad}}$
$z^2 + x^3 + y^5$		E_8		$p \neq 2,3,5$	E_8

Deformations of rational double points in nilpotent varieties of simple Lie algebras

Definition. *The* adjoint representation *of G is*
$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \quad x \mapsto \mathrm{d}(\mathrm{Int}_x),$$
where $\mathrm{d}(\mathrm{Int}_x)$ is the differential of conjugation with x. The quotient $\gamma : \mathfrak{g} \to \mathfrak{g}/G$ by this action is called adjoint quotient *and $\mathscr{N}(\mathfrak{g}) := \gamma^{-1}(0)$ is the* nilpotent variety.

Let $x \in \mathfrak{g}$ and $Z_G(x) := \{g \in G | \mathrm{Ad}(g)(x) = x\}$. We call x
- regular *if $\dim Z_G(x) = \mathrm{rank}(G)$, and*
- subregular *if $\dim Z_G(x) = \mathrm{rank}(G) + 2$.*

A transverse slice *in $\mathfrak{g}$ to the orbit of $x \in \mathfrak{g}$ is a locally closed subvariety $S \subseteq \mathfrak{g}$ such that $x \in S$, $G \times S \to \mathfrak{g}$ is smooth and $\dim S$ is minimal.*

Theorem 1 ([2], [5], [3]). *Let G be of type Γ, $\mathrm{char}(k)$ very good for Γ and S a transverse slice to the orbit of a subregular, nilpotent $x \in \mathfrak{g}$ such that $S \cap Gx = \{x\}$. Then,*
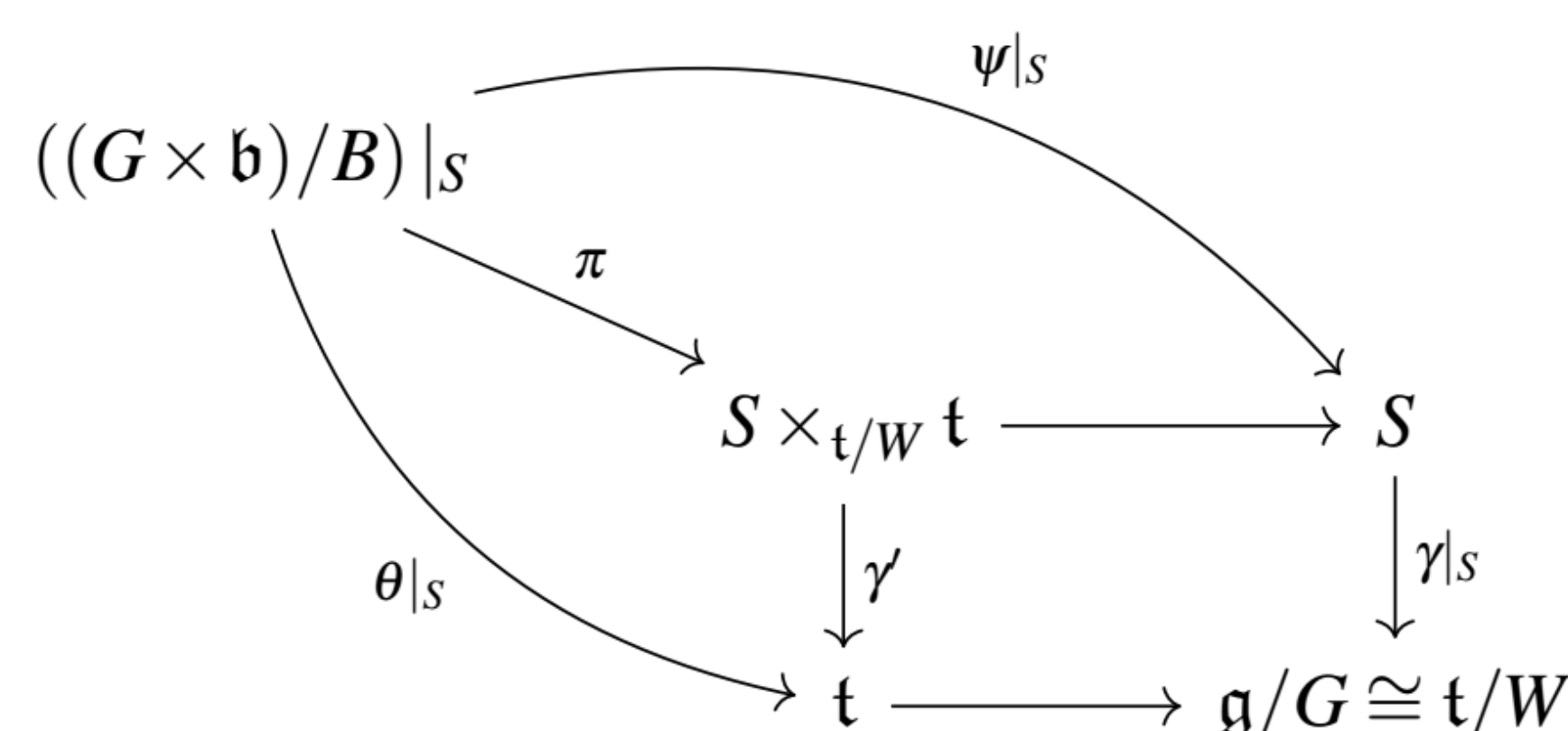- $(\mathscr{N}(\mathfrak{g}) \cap S, x)$ *is a rational double point of type Γ, and*
- $\gamma|_S$ *induces the miniversal deformation of $(\mathscr{N}(\mathfrak{g}) \cap S, x)$.*

Simultaneous resolutions

Theorem 2 (Grothendieck's simultaneous resolution of the adjoint quotient, [4]). *Under the assumptions of Theorem 1, there is a commutative diagram with maps*
$$\psi|_S : (g,b) \mapsto \mathrm{Ad}(g)(b)$$
$$\theta|_S : (g,b) \mapsto b_{\mathrm{ss}}$$
such that π is a simultaneous resolution of γ'.

Questions

- How can we generalize the above correspondence to all characteristics?
- Can we compute invariants of rational double points and their deformations using the connection with algebraic groups?

References

[1] M. Artin. "Coverings of the Rational Double Points in Characteristic p". In: *Complex Analysis and Algebraic Geometry*. Ed. by W. L. Baily and T. Shioda. Cambridge University Press, 1977, pp. 11–22.

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[4] P. Slodowy. *Four Lectures on Simple Groups and Singularities*. Vol. 11. Communications of the Mathematical Institute. Rijksuniversiteit Utrecht, 1980.

[5] P. Slodowy. *Simple Singularities and Simple Algebraic Groups*. Vol. 815. Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg, 1980.

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