Two discretizations of Koenigs nets and their connection

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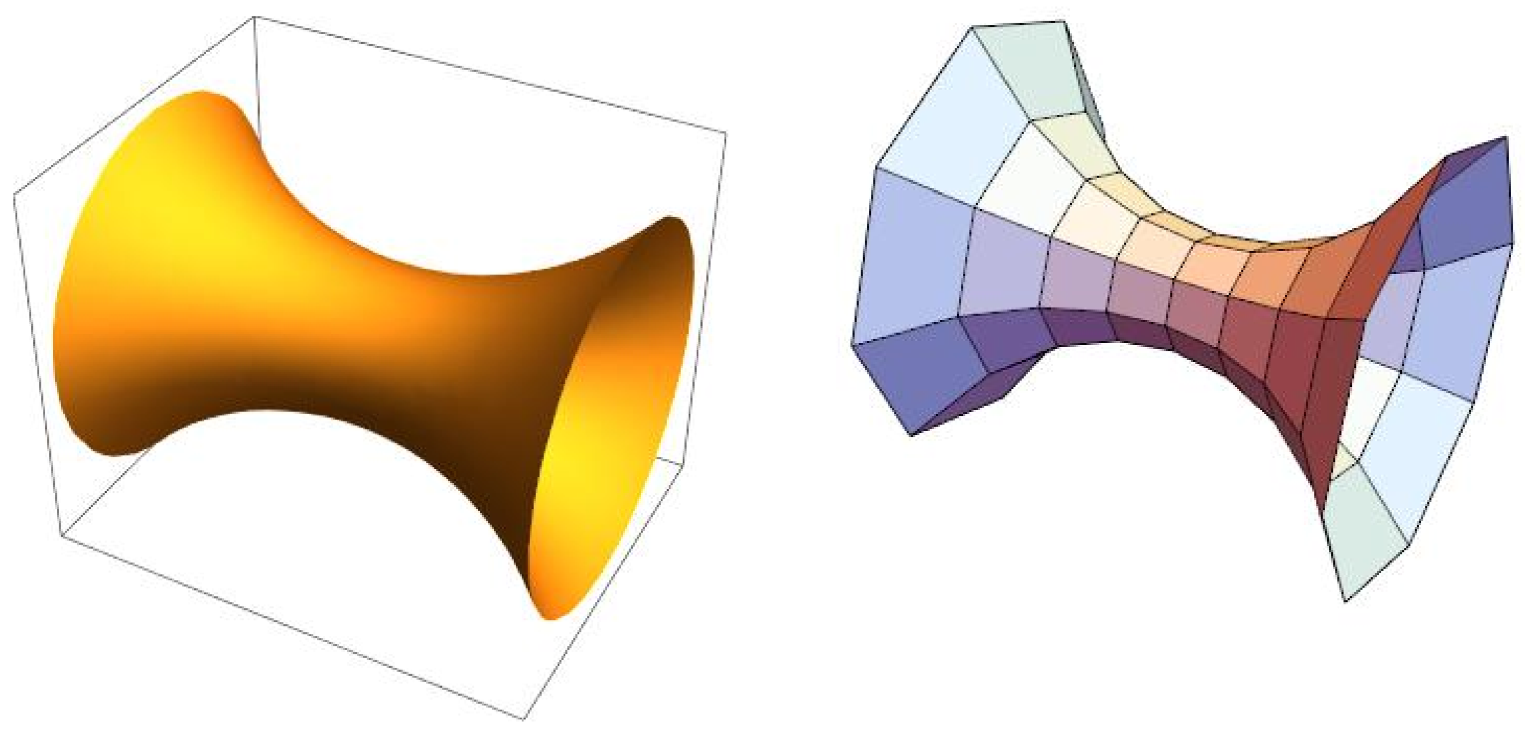
Abstract

We investigate the relation between two discretizations of Koenigs nets: The classical discretization of Bobenko and Suris, which defines discrete two-dimensional Koenigs nets as nets where the intersection points of diagonals build a net of planar quadrilaterals, and Doliwa's discretization, where a Koenigs lattice is defined as net which has six of its Laplace transforms on a conic at each quadrilateral. We prove that the nets defined by intersection points of diagonals of a classical discretized two-dimensional Koenigs net are exactly Doliwa's lattices. We describe how a classical Koenigs net can be constructed on a Doliwa lattice.

From smooth to discrete



1. Smooth nets are maps $f : \mathbb{R}^m \to \mathbb{R}^N$
2. Discrete nets are maps $f : \mathbb{Z}^m \to \mathbb{R}^N$

A large topic of discrete differential geometry is the discretization of special classes of nets.

Smooth Koenigs nets

We are concerned with the class of Koenigs nets.

Definition 1. Koenigs net A map $f : \mathbb{R}^2 \to \mathbb{R}^N$ is called a Koenigs net if there exists a scalar function $\nu : \mathbb{R}^2 \to \mathbb{R}_+$, such that

$$\partial_1\partial_2 f = (\partial_2 \log \nu)\partial_1 f + (\partial_1 \log \nu)\partial_2 f \quad (1)$$

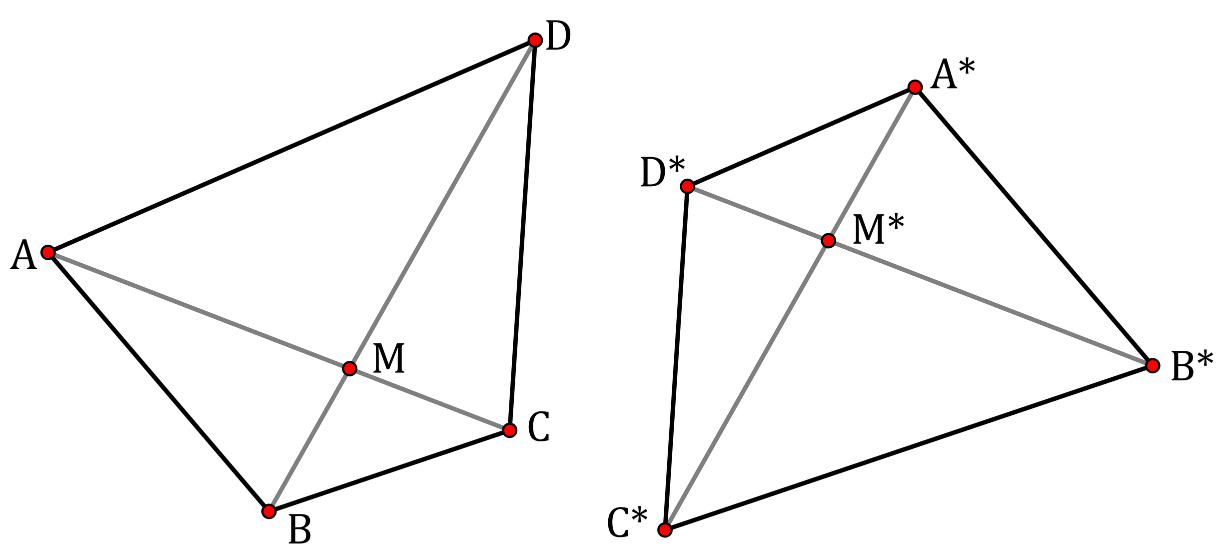
From this definition one can find multiple equivalent characterizations. Two of them have been discretized to obtain discrete nets with similar properties as the smooth net:

1. Koenigs nets can be found to be nets which admit a *dual*. Bobenko and Suris discretize the notion of duality to define the *classical discrete Koenigs net*.
2. Koenigs nets can be defined as nets which have their *Laplace transforms* in second order contact with a conic. Doliwa discretizes this property to define the *Koenigs lattice*.

Note that the discretizations are not equivalent. We are interested in their connection.

Classical discretization of Koenigs nets

Bobenko and Suris discretize duality:
Two planar quadrilaterals are called dual if their corresponding edges are parallel and their noncorresponding diagonals are parallel.



Dual quadrilaterals

Definition 2 (Discrete Koenigs net)**.** A net $f : \mathbb{Z}^m \to \mathbb{R}^N$ with planar quadrilaterals is called a discrete Koenigs net if it admits a dual, i.e. a net $f^* : \mathbb{Z}^m \to \mathbb{R}^N$ with planar quadrilaterals such that all elementary quadrilaterals of the net f^* are dual to the corresponding quadrilaterals of f.

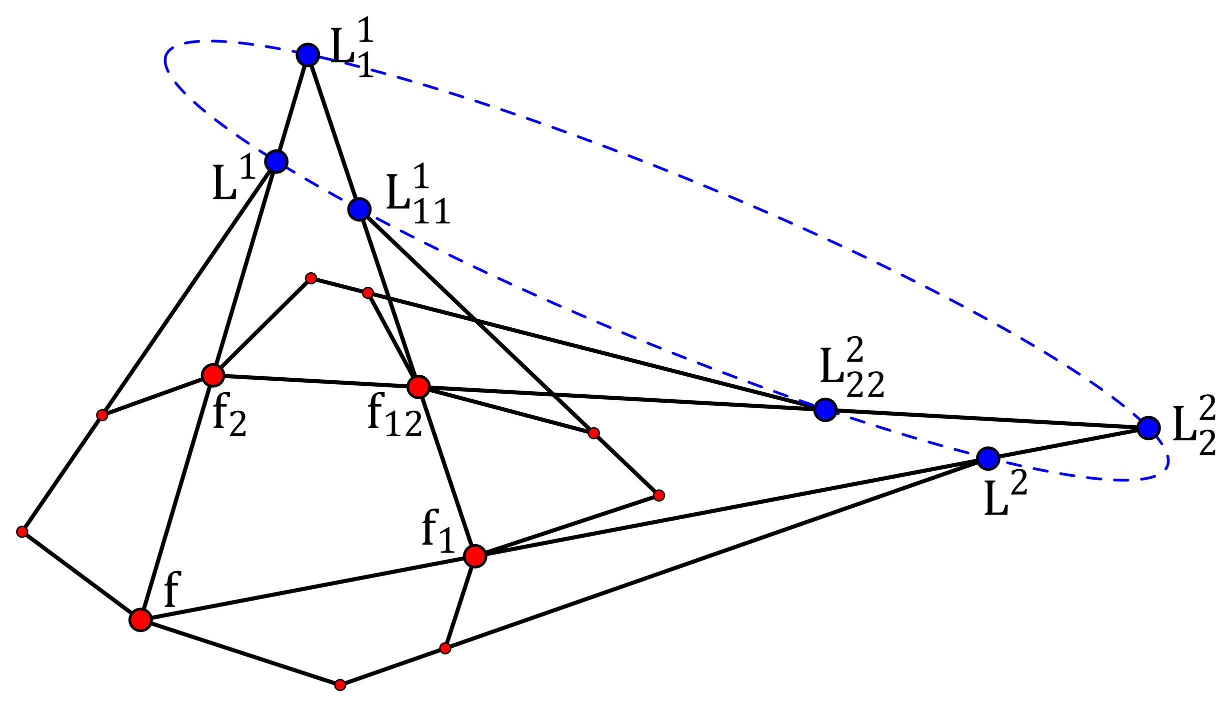
One can prove a geometric characterization.

Theorem 3 (Geometric characterization)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$ be a net with planar quadrilaterals such that for every point $f = f(u)$ its four neighbours $f_{\pm1}$, $f_{\pm2}$ are not coplanar. Then f is a discrete Koenigs net if and only if for every point $f = f(u)$ the intersection points of diagonals of the four quadrilaterals adjacent to f are coplanar, that is, if the intersection points of diagonals build a new net with planar quadrilaterals.*

Doliwa's discretization

The discrete Laplace transforms of a discrete net with planar quadrilaterals can be defined as the intersection points of the opposite edges of each quadrilateral.

Definition 4 (Doliwas Koenigs lattice)**.** Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$ be a net with planar quadrilaterals. It is called a Koenigs lattice if for every quadrilateral of the net there exists a conic passing through the six Laplace transforms L^1, L^1_1, L^1_{11}, L^2, L^2_2, L^2_{22}.

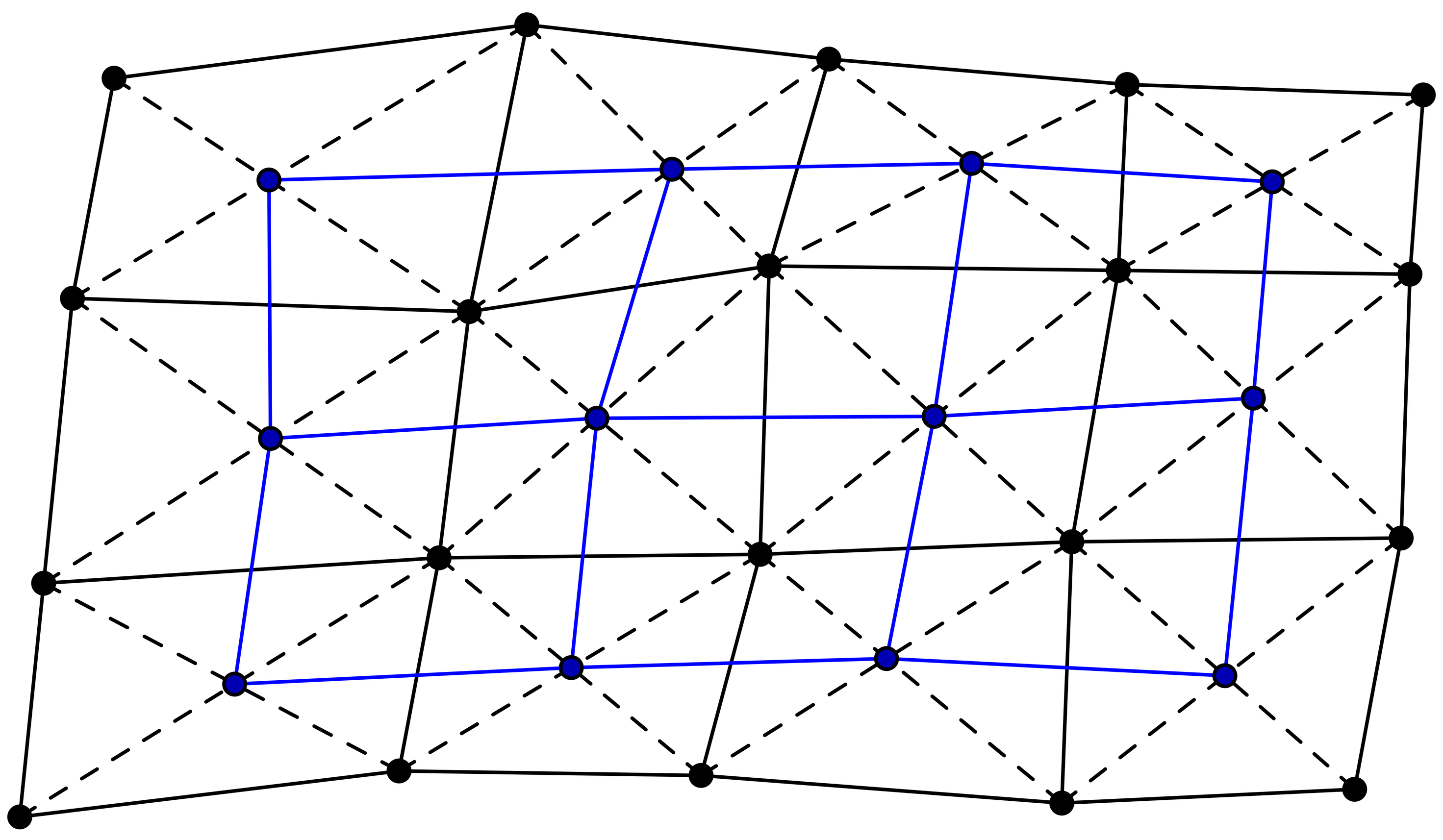


The six Laplace transforms at a quadrilateral $(f f_1 f_{12} f_2)$

Connection of both discretizations

There is a simple connection between both characterizations:

Theorem 5. *A map $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is a Koenigs lattice of Doliwa if and only if a classical Koenigs net $g : \mathbb{Z}^2 \to \mathbb{R}^N$ exists such that f consists of the intersection points of diagonals of g*



A discrete Koenigs net(black) and its intersection points of diagonals, which are exactly Doliwa's Koenigs lattices(blue)

In the proof we show how such a Koenigs net g can be constructed on a given Koenigs lattice f. It turns out that there is a freedom of $2N + 2$, i.e. there exists a $(2N + 2)$-parameter family of Koenigs nets belonging to one Koenigs lattice.

References

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2. A. Bobenko and Y. Suris. "Discrete Koenigs nets and discrete isothermic surfaces". In: *Intern. Math. Research Notices* 11 (2009), pp. $1976 - 2012$.
3. A. Doliwa. "Geometric discretization of the Koenigs nets". In: *J. Math. Phys.* 44 (2003), pp. $2234 - 2249$.

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