Ground state transition in certain models of finite rank perturbations of random potentials

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Abstract

The limiting behaviour of the ground state for certain models of finite rank perturbations of finite dimensional random diagonal potentials is considered. Finite rank perturbation theory is used to reduce the problem to the spectral properties of a meromorphic matrix valued function of fixed dimension, which is then solvable by regularisation and restriction to appropriate subsets of the probability space. By these means the existence of two regimes for the parameter of disorder is shown. The two regimes are characterised by localisation respectively delocalisation of the ground state. As part of the result precise asymptotics on the ground energy and on a norm-quotient characterising localisation are given. Additional results include a detailed description of the behaviour of the matrix-valued function's eigenvalues and a resulting statement on the extent of the whole spectrum of the model.

the Model

the model is a sequence of matrices of the following form:

$$H_N := H_N[\omega] = \underbrace{\kappa_N \text{diag}\left(V(1)[\omega], \ldots, V(N)[\omega]\right)}_{=:D_N} - T_N$$

where $\left(V(n)\right)_{n\in\mathbb{N}}$ are i.i.d. $\mathcal{N}(0,1)$ variables and

$$T_N = \sum_{j=1}^{K} \alpha_j \left|\phi_j^N\right\rangle \left\langle \phi_j^N\right|$$

cyclic, with $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$, $(\phi_j^N)_{j\leq N}$ orthonormal and

$$\kappa_N = \lambda / \sqrt{2\log N}$$

a normalising constant, which guarantees that the spectrum of D_N typically spans the interval $[-\lambda, \lambda]$. Furthermore a delocalisation assumption for T_N of the following form is made:

$$\exists \delta > 0 \text{ s. t. } \forall k \in K : \|\phi_k^N\|_4^4 \in \mathrm{O}\left(N^{-\delta}\right)$$

Results

the model exhibits a ground state transition at $\lambda = 1$, characterised by a change of the localisation behaviour of the ground state and the asymptotics of its energy.

Theorem 1. *Assuming delocalisation of T_N, the ground energy E_1^N and corresponding ground state ψ_1^N satisfy with asymptotically full probability the following asymptotics: if $\lambda < 1$:*

$$E_1 = -1 - \kappa_N^2 - \mathrm{O}\left(\kappa_N^4\right)$$

$$\Omega\left(N^{\delta/4}\right) \leq \frac{\|\psi_1^N\|_2}{\|\psi_1^N\|_\infty} \leq \mathrm{O}\left(N^{\delta/2}\right)$$

if $\lambda > 1$:

$$\frac{\|\psi_1^N\|_2}{\|\psi_1^N\|_\infty} = 1 + \mathrm{O}\left(N^{-\delta/4}\right)$$

$$E_1 = \min_{x\in N} \kappa_N V(x) + \mathrm{O}\left(N^{-\delta/2}\right)$$

where a property $A(N,\omega)$ is said to hold with *asymptotically full probability (a. f. p.)* if

$$\mathbb{P}\left(\{\omega \in \Omega \mid A(N,\omega) gilt\}\right) \xrightarrow{N\to\infty} 1$$

in that sense the Landau notation is to be understood as:

$$X_N \in \mathrm{O}\left(g(N)\right) :\iff \exists C \in \mathbb{R} \text{ mit: } \mathbb{P}\left(X_N > Cg(N)\right) \xrightarrow{N\to\infty} 0$$

analogously for $\mathrm{o}(\cdot), \Omega(\cdot)$ etc., i.e. the Landau-Notation holds with a. f. p. with constant C.

Extent of $\sigma(H_N)$

by analysing the behaviour of the eigenvalues of M_N^z it follows that the spectrum of D_N and H_N are intertwined in the sense that the empirical distribution functions of their eigenvalues satisfy $N(D_N, x) \leq N(H_N, x) \leq N(D_N, x) + K$ this implies that the asymptotic extent of $\sigma(D_N)$ transfers to H_N i.e.

$$\sigma(H_N) \setminus \{E_1, \cdots, E_K\} \xrightarrow[N\to\infty]{H} [-\lambda, \lambda] \text{ in probability} \qquad (1)$$

where H is the Hausdorff-distance of compact sets.

Finite Rank Perturbation Theory

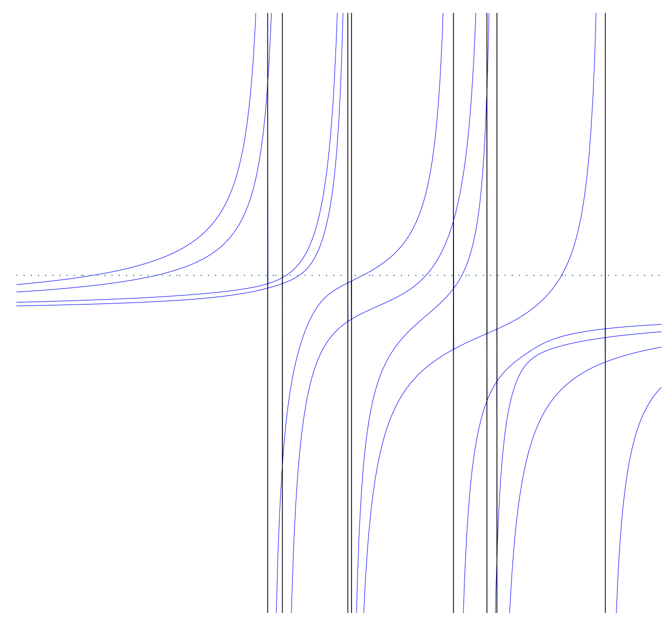
By the means of finite rank perturbation theory the question of the ground state and function can be reduced to the spectral properties of a (random) matrix valued meromorphic function M_N^z: $\mathbb{C} \to \mathbb{C}^{K\times K}$ with

$$(M_N^z)_{i,j} := \sum_{x\in[N]} \frac{\sqrt{\alpha_i\alpha_j}}{\kappa_N V(x) - z} \overline{\phi_i^N(x)} \phi_j^N(x)$$

where the following identity can be shown:

$$E \in \sigma(H_N) \iff 1 \in \sigma(M_N^E)$$

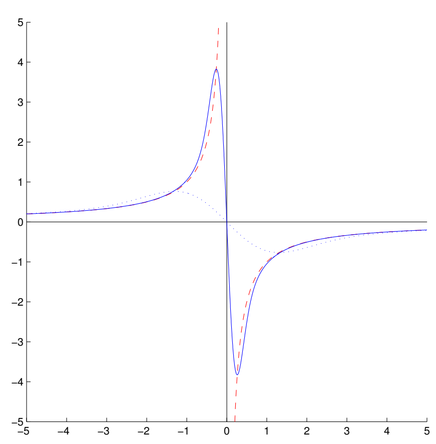
The eigenvalues $\lambda_k(z)$ of M_N^z are well behaved as they are meromorphic in z and (uniformly in the disorder) Lipschitz-continuous away from their singularities.



a typical realisation of the eigenvalues $\lambda_k(z)$(blue lines), the potential values $\kappa_N V(n)$ (black lines) and the line y=1 (dotted line) for the parameters $K = 4$ and $N = 8$

Probabilistic Considerations

By making multiple assumptions which hold with a. f. p. the uniform(in z and ω) convergence of $\bar{M}_N^z$, a regularised version of M_N^z, towards $\rho_N(z)\text{diag}(\alpha_1, \cdots, \alpha_K)$ on apropriate compact subsets can be shown. Here ρ_N is up to a scaling the Hilbert transform of the $\mathcal{N}(0,1)$ density and it converges uniformly on compact $C \subset \mathbb{R} \setminus \{0\}$ to $-\frac{1}{x}$



$\rho_N(z)$ for $\kappa_N = 1$(dotted line), $\kappa_N = 0.1$ and the function $-\frac{1}{z}$(dashed line)

In the case $\lambda < 1$ the regularisation captures already with a. f. p. all point of M_N^z and the statements follow easily. For $\lambda > 1$ different behaviour is expected as the lowest potential value is close to E_0 and does not fall within the regularisation. It holds that

$$M_N^z = \bar{M}_N^{z,\epsilon} + S_N^z + \mathrm{o}(1)$$

where S_N^z is the contribution of $\arg\min_{x\in N} V(x)[\omega]$. The resulting necessity of S_N^z of being of constant order then yields the respective statements of the theorem.

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