Generalization of Kantorovich Duality to multi-marginal optimal transportation and applications in economics

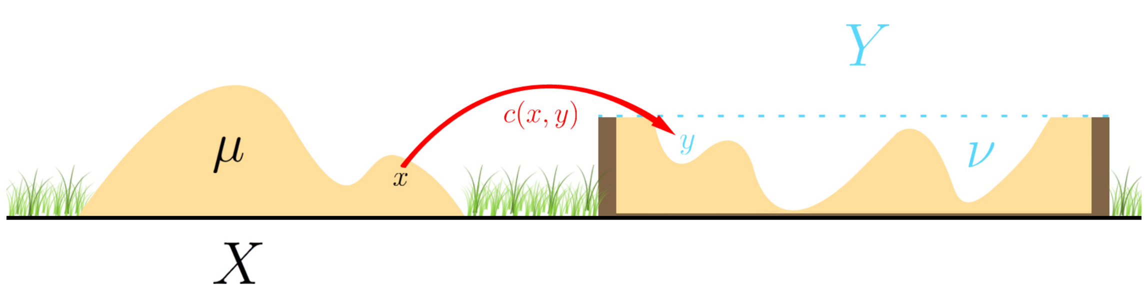
Daniela Vögler, Prof. Dr. Gero Friesecke
Center for Mathematical Sciences, Technische Universität München
Chair of Analysis

Abstract

To transport something in an optimal manner - this is a problem people faced for hundreds and thousands of years. But just recently - in the 1940s - Kantorovich tried to find a solution to this problem by modeling it from a mathematical point of view. He formulated 'Kantorovich's optimal transportation problem'. As a linear minimization problem with convex constraints, it admits a dual formulation which is celebrated as the Kantorovich duality. However, he only focused on the two marginal setting. The generalization of this famous duality to multi-marginal optimal transportation creates far more ways to apply this topic in economics.

Theory of optimal transportation

It is a common occurrence that children have the great idea to build their sandcastles in the grass instead of the sandbox. Then their parents face the task of restoring the initial situation in the most efficient manner possible. This problem is the perfect approach to introduce the theory of optimal transportation. Both the pile and the hole in the sandbox have the same volume which can be normalized to 1. Hence, they can be modeled by probability measures μ and ν defined respectively on some measure spaces X and Y. The energy consumed by moving the sand around is represented by the measurable and non-negative cost function $c\colon X \times Y \to \mathbb{R}_0^+ \cup \{+\infty\}$.



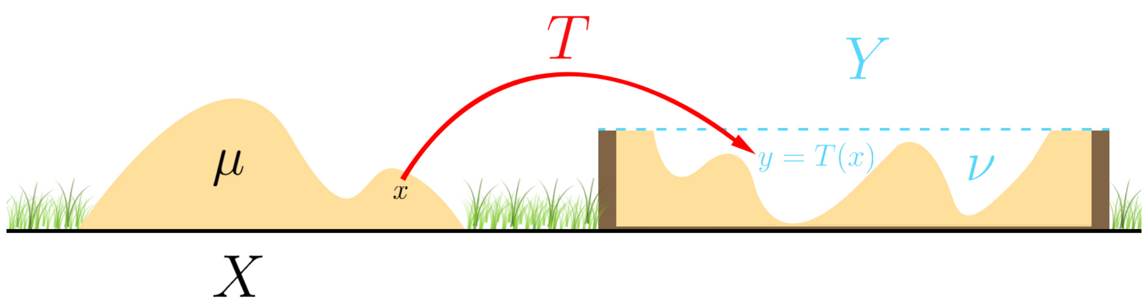
Historically Monge presented the initial approach. He captured the given problem in the following formulation:

Monge's optimal transportation problem

Minimize

$$I[T] = \int_X c(x, T(x)) d\mu(x)$$

over the set of all measurable maps T such that $T\#\mu = \nu$.



Kantorovich built upon Monge's results, however he changed one vital aspect. He left himself a loophole enabling him to split mass by using probability measures on the product space $X \times Y$ as transference plans. All of the measures on the product space are summarized in the set $\mathcal{P}(X \times Y)$. As it is necessary that μ is transported to ν, only those transference plans are admissible, which are an element of

$$\Pi(\mu,\nu) = \{\pi \in \mathcal{P}(X \times Y) | \pi(A \times Y) = \mu(A), \pi(X \times B) = \nu(B)$$

$$\text{for all measurable } A \subseteq X \text{ and } B \subseteq Y\}.$$

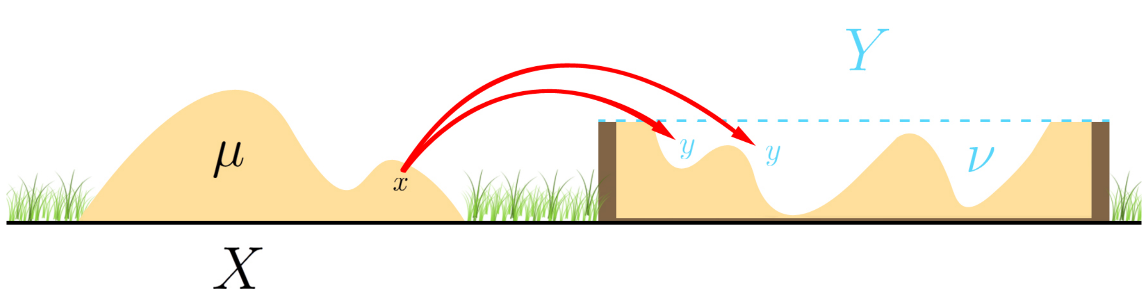
This defintion at hand one can move onto considering:

Kantorovich's optimal transportation problem

Minimize

$$I[\pi] = \int_{X\times Y} c(x,y) d\pi(x,y)$$

for $\pi \in \Pi(\mu,\nu)$.



Kantorovich's optimal transportation problem is a linear minimization problem with convex constraints. Therefore it admits a dual formulation. In the considered case this formulation is the celebrated Kantorovich duality. As the optimal transportation problem itself can be transferred in the multi-marginal setting in a natural manner, one is tempted to translate the dual problem as well. Then the multi-marginal duality provides an important alternative formulation to the transportation problem. Instead of minimizing the functional I over all probability measures with fixed marginals, one can maximize a certain dual problem over n potentials, each of which is weighted with one of the marginals.

Multi-marginal optimal transportation

Theorem: Generalized Kantorovich Duality

Let $X_1, ..., X_n$ be Polish spaces and $\mu_1, ..., \mu_n$ be Borel-probability measures on $X_1, ..., X_n$ respectively. Moreover, assume that $c : X_1 \times ... \times X_n \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semi-continuous cost function.

Now we define

- $\forall \pi \in \mathcal{P}(X_1 \times ... \times X_n) : I[\pi] = \int_{X_1\times...\times X_n} c(x_1, ..., x_n) d\pi(x_1, ..., x_n)$,
- $\forall (\varphi_1, ..., \varphi_n) \in L^1(d\mu_1) \times ... \times L^1(d\mu_n) : J(\varphi_1, ..., \varphi_n) = \sum_{i=1}^n \int_{X_i} \varphi_i(x_i) d\mu_i$,
- $\Pi(\mu_1, ..., \mu_n)$ to be the set of all Borel probability measures on $X_1 \times ... \times X_n$ with marginals μ_1 on X_1,..., μ_n on X_n,
- the set of all measurable functions $(\varphi_1, ..., \varphi_n) \in L^1(d\mu_1) \times ... \times L^1(d\mu_n)$ satisfying a certain upper boundary condition almost everywhere:

$$\Phi_c \cap L^1 = \{(\varphi_1, ..., \varphi_n) \in L^1(d\mu_1) \times ... \times L^1(d\mu_n) | \sum_{i=1}^n \varphi_i(x_i) \le c(x_1, .., x_n)$$

$$\textit{for } d\mu_1 - \textit{almost all } x_1 \in X_1, ..., d\mu_n - \textit{almost all } x_n \in X_n\}.$$

Then

$$\inf_{\pi\in\Pi(\mu_1,...,\mu_n)} I[\pi] = \sup_{(\varphi_1,...,\varphi_n)\in\Phi_c\cap L^1} J(\varphi_1, ..., \varphi_n)$$

holds.

Moreover, the infimum on the left-hand side is attained, i.e. there is a $\pi^* \in \Pi(\mu_1, ..., \mu_n)$, such that $\inf_{\pi\in\Pi(\mu_1,...,\mu_n)} I[\pi] = I[\pi^*]$.

Furthermore,

$$\sup_{(\varphi_1,...,\varphi_n)\in\Phi_c\cap L^1} J(\varphi_1, ..., \varphi_n) = \sup_{(\varphi_1,...,\varphi_n)\in\Phi_c\cap C_b} J(\varphi_1, ..., \varphi_n)$$

holds. Thereby, the definition of $\Phi_c \cap C_b$ naturally appears by inspecting the definition of $\Phi_c \cap L^1$.

Economic applications

The generalized Kantorovich duality is a great tool to tackle economic problems. In the following the market for houses is captured within the mathematical objects of the Kantorovich duality.

mathematical formulation	economic interpretation
$i \in [n]$	This is one of the n goods, which are necessary in order to build a house.
X_i for $i \in [n]$	This is a quality space containing all the qualities the good i is available in.
μ_i for $i \in [n]$	This is the distribution of the quality space, which is determined by the wealth and behaviour of the costumers.
$c : X_1 \times ... \times X_n \to \mathbb{R}_+ \cup \{+\infty\}$	$c(x_1, ..., x_n)$ are the expenses of a house-building-company (hbc) if they build a house of the quality $(x_1, ..., x_n)$.
$\Pi(\mu_1, ..., \mu_n)$	These are all the possibilties for the hbc to meet the demand.
$I[\pi] = \int_{X_1\times...\times X_n} c(x_1, ..., x_n) d\pi(x_1, ..., x_n)$ for $\pi \in \Pi(\mu_1, ..., \mu_n)$	These are the expenses of the hbc if it meets the costumers demand, following $\pi \in \Pi(\mu_1, ..., \mu_n)$.
$\inf_{\pi\in\Pi(\mu_1,...,\mu_n)} \int_{X_1\times...\times X_n} c(x_1, ..., x_n) d\pi(x_1, ..., x_n)$	These are the minimal costs the hbc has to pay in order to meet the demand.
$\varphi_i \colon X_i \to \mathbb{R}$ für $i \in [n]$	These are the costs the hbc has to pay in order to buy the good i in quality x_i from a third party.
$\forall x_1 \in X_1, ..., x_n \in X_n : \sum_{i=1}^n \varphi_i(x_i) \le c(x_1, ..., x_n)$	The third party arranges its prices in such a manner, that hbc has to pay at most the in-house production costs.
$\inf_{\pi\in\Pi(\mu_1,...,\mu_n)} I[\pi] = \sup_{(\varphi_1,...,\varphi_n)\in\Phi_c\cap L^1} J(\varphi_1, ..., \varphi_n)$	The third party can arrange its price functions in such a way that if the hbc accepts the deal, it would have to pay (almost) as much as it would have ideally paid producing the goods itself.

References

1. Guillaume Carlier. *Transportation and Economic Applications*. www.ceremade.dauphine.fr/~carlier/IMA-transport-Lecture-Notes.pdf, June 2010, Lecture Notes.
2. Cédric Villani. *Topics in Optimal Transportation*. American Mathematical Soc., Rhode Island, 2003.