Circle packings with tangential boundary circles for numerical conformal mapping

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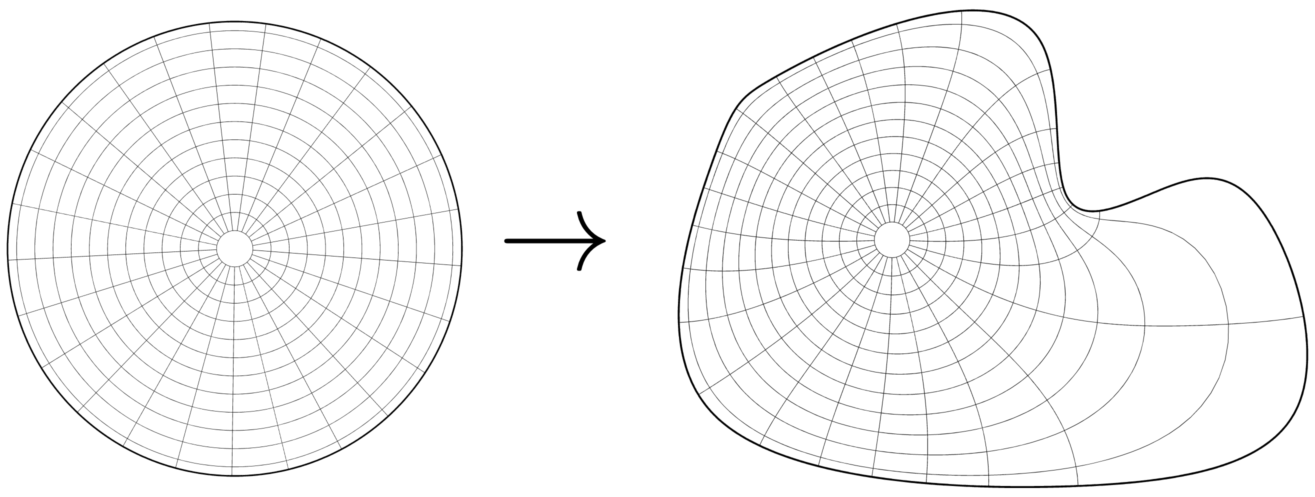
Abstract

The Riemann Mapping theorem ensures the existence of conformal maps between arbitrary simply connected plane domains, however it is not much aid in actually computing them. A relativley new algorithm based on ideas of W. Thurston approximates these using circle packings. This algorithm has been proven to converge, but the convergence is very slow and not uniform. The reason for this is that the approximation is very inexact near the boundary. This poster illustrates an algorithm for computing circle packings on both domains, whose boundary circles are tangent to the domain boundary. Computing these packings rests on a Newton method.

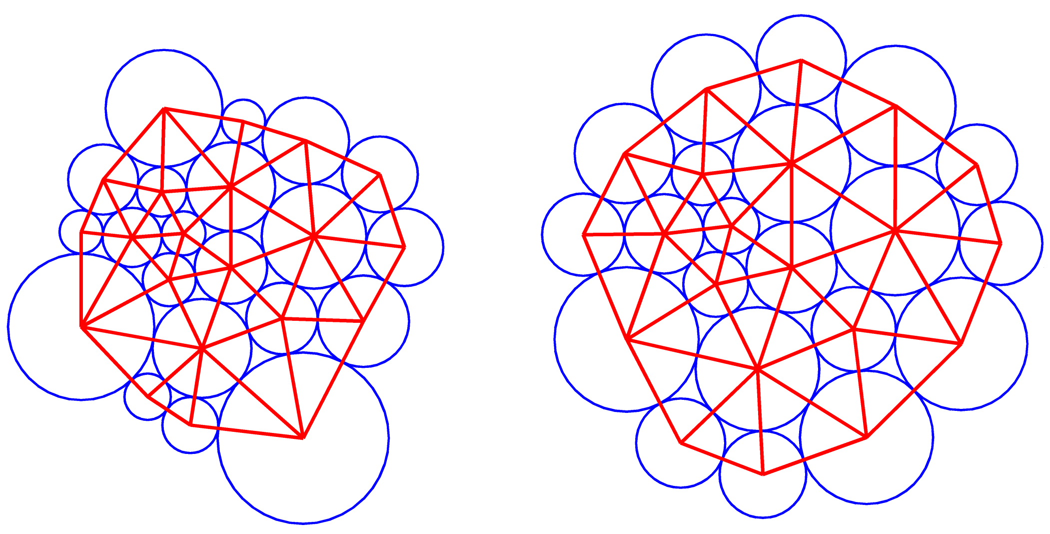
The Problem

Given $\Omega \subsetneq \mathbb{C}$ simply connected.
Find $f : \Omega \to \mathbb{D} = \{z \in \mathbb{C} \,||z| < 1\}$ bijective and conformal. E.g.:



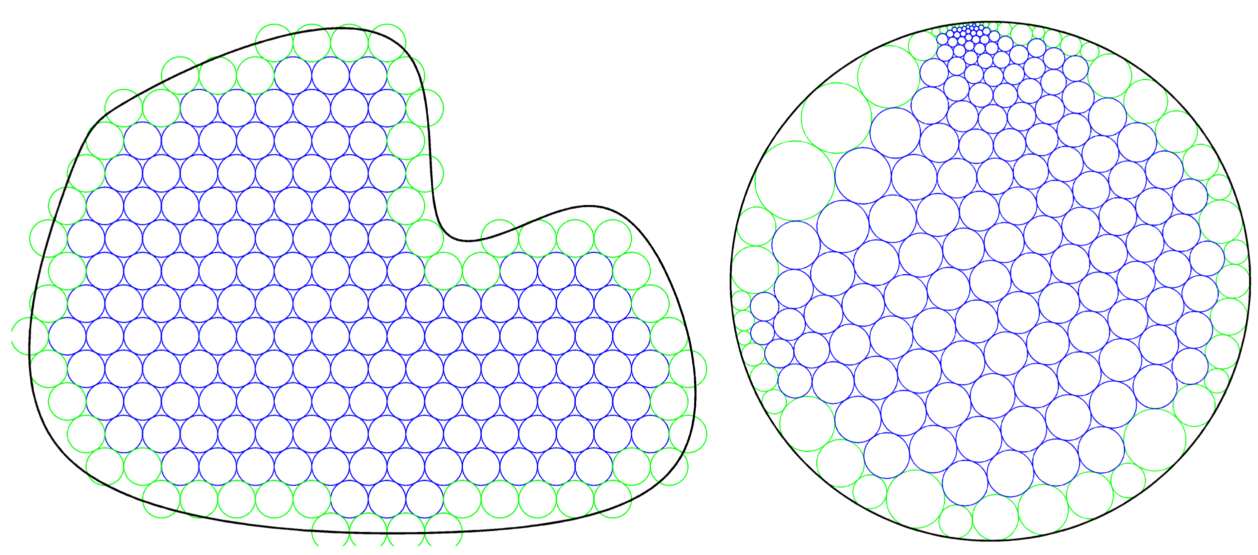
The classical circle packing method

Idea: Conformal maps map infinitely small circles to infinitely small circles
$\Rightarrow$ Approximate them with maps that map actual circles to actual circles!
For this, we use circle packings, i.e. configurations of mutually tangent circles. For example:



The red graph indicates their *tangency structure*: Tangent circles correspond to adjacent vertices.

Idea: Understand the relation between packings with the *same underlying graph* as discrete conformal maps. $\Rightarrow$ Approximate a map, by covering the two domain Ω roughly with circle packings with the same tangencies:



$\rightarrow$ How to compute these maps?
We can cover Ω only roughly (as seen above) and then compute a packing with tangential boundary circles on $\mathbb{D}$. This can be done rather simply using hyperbolic geometry.

However: This results in a bad approximation!
For a better sapproximation, we would like to have packings with tangential boundary circles on *both* domains.
$\Rightarrow$ How to compute these?

References

1. Stephenspn, K.: Introduction to circle packing, Cambridge University Press, Cambridge, MR2131318 (2006a:52022) (2005).
2. Wegert, E., Roth, O., Kraus, D., On Beurling's boundary value problem in circle packing, Complex Var. Elliptic Equ. 57 (2012), no. 2-4, 397-410. MR2886749
3. Volland, D., Circle packings with tangential boundary circles for numerical conformal mapping, Bachelor's Thesis (2014)

Newton Method

Idea: apply Newton's Method. $\Rightarrow$ Describe packings by variables and nonlinear equations.
Let n be the total number of circles, m the number of boundary circles in a packing.
$\rightarrow$ **Which variables describe our packing?**

- Any interior circle can be described via its radius and its center (3 real variables)
- Any boundary circle can be described via its radius and its contact point to the boundary (2 real variables)

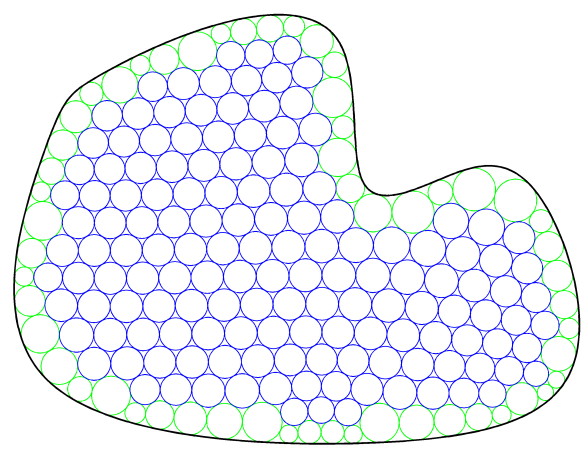
$\Rightarrow 3n - m$ **Variables**
$\rightarrow$ **Which equations do these have to fulfill?**

- Two circles (z_j, r_j), (z_k, r_k) must be tangent if required in the tangency graph:

$$|z_j - z_k| = r_j + r_k \tag{1}$$

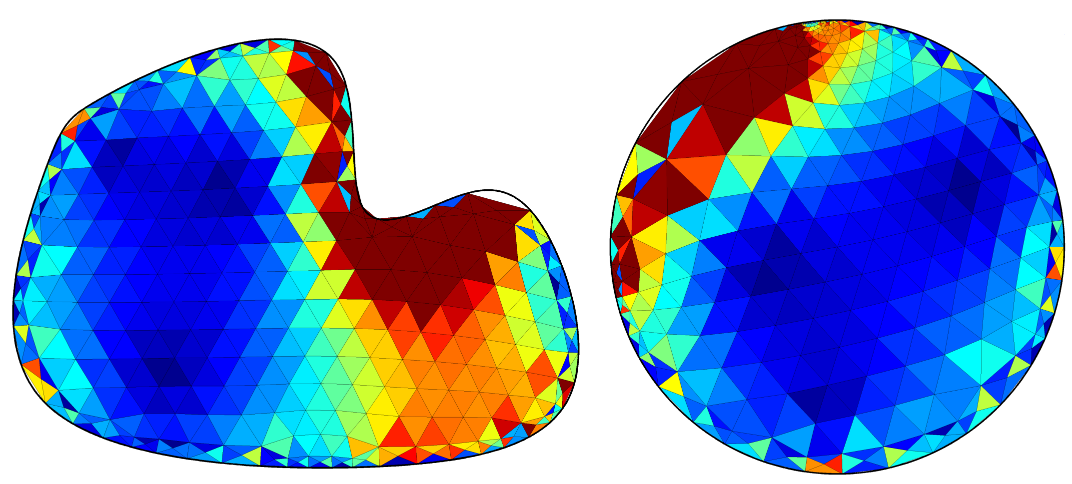
  By Euler's formula, there are $3n - m - 3$ tangencies.
- Additionally there are 3 normalization conditions.

So, we have a $3n - m$-dimensional root finding problem.
Packings from the classical method can be used as intial values. Applying Newton's method then for instance yields the following packing:

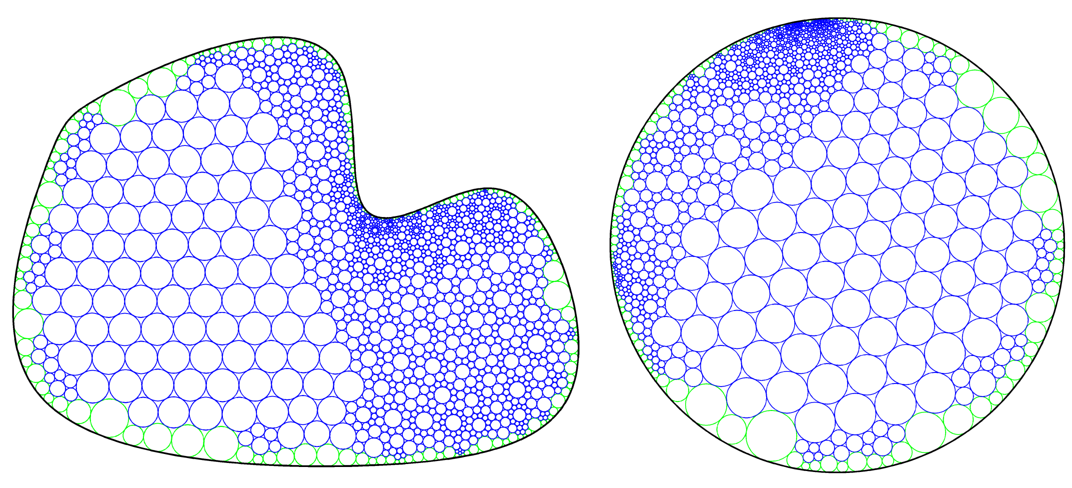


Refinement and Adaptivity

To obtain a continuous map from our discrete maps, we can map the triangles formed by the tangency graph onto each other using affine transformations.



Usually, these affine maps are *not* conformal: they map non-congruent triangles to each other, so they cannot preserve angles, i.e. be conformal.
We wish to approximate a conformal map, so the distrotion of angles caused by this is our *local error*. The plot above is colored accordingly.
The error varies across the domain $\Rightarrow$ exploit this with local/adaptive refinement!
$\rightarrow$ E.g., after several steps, we obtain the following refined packings:



Practical tests have shown that the maps obtained under refinement apparently converge to a classical conformal map as desired.