Multi-Population Phase Oscillator Networks with Higher-Order Interactions

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Abstract

The classical Kuramoto model consists of finitely many pairwise coupled oscillators on the circle. In many applications a simple pairwise coupling is not sufficient to describe real-world phenomena as higher-order (or group) interactions take place. Hence, we replace the classical coupling law with a very general coupling function involving higher-order terms. Furthermore, we allow for multiple populations of oscillators interacting with each other through a very general law. In our analysis, we focus on the characteristic system and the mean-field limit of this generalized class of Kuramoto models, in which the number of oscillators tends to infinity. We propose a general framework to work with three aspects (higher-order, multi-population, and mean-field) simultaneously. Assuming identical oscillators in each population, we derive equations for the evolution of oscillator populations in the mean-field limit. Then, we investigate dynamical properties within the framework of the characteristic system such as the stability of the state, in which all oscillators are synchronized within each population. Even though it turns out that this so called all-synchronized state is never asymptotically stable, under some conditions on the coupling function and with a suitable definition of stability, the all-synchronized state can be proven to be at least locally stable.

Keywords: Kuramoto model, Mean-field, Synchronization.

Mathematical Setting

Let $\mathcal{P}(\mathbb{S})$ denote the set of probability measures on the unit circle $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$. To compare two measures $\mu, \nu \in \mathcal{P}(\mathbb{S})$, we use the Wasserstein-1 distance [1], which is also referred to as the bounded-Lipschitz distance

$$W_1(\mu,\nu) := \inf_{\substack{\gamma\in\mathcal{P}(\mathbb{S}\times\mathbb{S})\\ M_1\gamma=\mu,\ M_2\gamma=\nu}} \int_{\mathbb{S}\times\mathbb{S}} |\alpha-\beta|_{\mathbb{S}}\ \gamma(\mathrm{d}\alpha,\mathrm{d}\beta) \tag{1a}$$

$$= \sup_{f\in\mathcal{D}} \left| \int_{\mathbb{S}} f(\alpha)\ \mathrm{d}\mu(\alpha) - \int_{\mathbb{S}} f(\alpha)\ \mathrm{d}\nu(\alpha) \right|, \tag{1b}$$

where $M_1\gamma$ and $M_2\gamma$ are the marginals of γ, i.e., the push-forward measures under the map $(\alpha,\beta)\mapsto\alpha$ and $(\alpha,\beta)\mapsto\beta$ and

$$\mathcal{D} := \{f \in C(\mathbb{S}) : |f(\alpha)-f(\beta)| \le |\alpha-\beta|_{\mathbb{S}} \text{ for all } \alpha,\beta\in\mathbb{S}\}.$$

Further, for $n \in \mathbb{N}$ we write $[n] := \{1,\dots,n\}$ and for $R\in\mathbb{N}$ we define the multi-index $s = (s_1,\dots,s_R) \in [M]^R$. Then, given $\mu = (\mu_1,\dots,\mu_M) \in \mathcal{P}(\mathbb{S})^M$, we define the measure

$$\mu^{(s)} = (\mu_{(s_1)},\dots,\mu_{(s_R)})$$

and write $|s| = R$.

The Equations

We consider the dynamics of $M \in \mathbb{N}$ coupled phase oscillator populations. We now introduce a general set of equations that describes the network evolution, where the state of population $\sigma \in [M]$ is given by a probability measure μ_σ.

The network interactions are determined by a multi-index $r^\sigma \in [M]^{R_\sigma}$ for each population together with Lipschitz continuous coupling functions $g_\sigma\colon \mathbb{S}^{|r^\sigma|}\times\mathbb{S}^{|r^\sigma|}\times\mathbb{S}\to\mathbb{R}$. Specifically, these coupling functions are supposed to be L-Lipschitz when $\mathbb{S}^{|r^\sigma|}\times\mathbb{S}^{|r^\sigma|}\times\mathbb{S}$ is considered with the metric $d(\alpha,\beta) = \sum_{i=1}^{2|r^\sigma|+1}|\alpha_i-\beta_i|_{\mathbb{S}}$. If $\mu^{\mathrm{in}} = (\mu_1^{\mathrm{in}},\dots,\mu_M^{\mathrm{in}}) \in \mathcal{P}(\mathbb{S})^M$ denotes the initial state of the network, $\#$ denotes the push-forward operator and $\mu = (\mu_1,\dots,\mu_M)$, then the evolution of $\mu(t) = (\mu_1(t),\dots,\mu_M(t))$ is determined by characteristic equations [2]

$$\partial_t\Phi_\sigma(t,\xi_\sigma^{\mathrm{in}},\mu^{\mathrm{in}}) = (\mathcal{K}_\sigma\mu(t))(\Phi_\sigma(t,\xi_\sigma^{\mathrm{in}},\mu^{\mathrm{in}})) \tag{2a}$$

$$\mu_\sigma(t) = \Phi_\sigma(t,\cdot,\mu^{\mathrm{in}})\#\mu_\sigma^{\mathrm{in}} \tag{2b}$$

$$\Phi_\sigma(0,\xi_\sigma^{\mathrm{in}},\mu^{\mathrm{in}}) = \xi_\sigma^{\mathrm{in}}. \tag{2c}$$

for $\sigma\in[M]$ and the evolution operator

$$(\mathcal{K}_\sigma\mu)(\phi) = \omega_\sigma + \int_{\mathbb{S}}\int_{\mathbb{S}^{|r^\sigma|}}\int_{\mathbb{S}^{|r^\sigma|}} g_\sigma(\alpha-\beta,\gamma-\phi)\ \mathrm{d}\mu^{(r^\sigma)}(\alpha)\mathrm{d}\mu^{(r^\sigma)}(\beta)\mathrm{d}\mu_\sigma(\gamma), \tag{3}$$

where $\omega_\sigma\in\mathbb{R}$ is the instantaneous frequency of all oscillators in population σ.

The Synchronized State

We define the set of **synchronized states** as $\mathrm{S} = \{\delta_\phi : \phi\in\mathbb{S}\}$. The set of **all-synchronized states** is then given by S^M.

Definition 1 *The set S^M is* **stable** *if for all $\sigma\in[M]$ and all neighborhoods $U_\sigma\subset\mathcal{P}(\mathbb{S})$ of S there exist neighborhoods V_σ of S such that for any $\mu^{\mathrm{in}} = (\mu_1^{\mathrm{in}},\dots,\mu_M^{\mathrm{in}}) \in V_1\times\dots\times V_M$, the solution $\mu(t)$ of* (2),(3) *satisfies $\mu(t)\in U_1\times\dots\times U_M$ for all $t\ge 0$.*

Theorem

Let

$$g_\sigma^{(0,1)}(\alpha,\gamma) := \frac{\partial}{\partial\gamma}g_\sigma(\alpha,\gamma), \qquad a_\sigma := g_\sigma^{(0,1)}(0,0).$$

Theorem 2 *If the coupling functions g_σ are continuously differentiable, i.e. $g_\sigma\in C^1(\mathbb{S}^{|r^\sigma|}\times\mathbb{S})$, and they satisfy $a_\sigma>0$ for all $\sigma\in[M]$ then, the set of all-synchronized states S^M is stable.*

A Simplified System

As a simplified case of system (2)–(3), we take $M=1$ and $s^1=\{\}$. Then, the system takes the form

$$\partial_t\Phi(t,\xi,\mu^{in}) = (\mathcal{K}\mu(t))(\Phi(t,\xi,\mu^{in})) \tag{4a}$$

$$\mu(t) = \Phi(t,\cdot,\mu^{in})\#\mu^{in} \tag{4b}$$

$$\Phi(0,\xi,\mu^{in}) = \xi, \tag{4c}$$

with coupling operator

$$(\mathcal{K}\mu)(\phi) = \omega + \int_{\mathbb{S}} g(\gamma-\phi)\ \mathrm{d}\mu(\gamma). \tag{5}$$

Idea of the Proof for One Population

In order to prove the theorem for the simplified system (4)–(5), we proceed as follows:

1. Aim: $W_1(\mathrm{S},\mu_t) < \epsilon_U$ for all $t\ge 0$. Choose $V = \mathbb{B}(\mathrm{S},\epsilon_V)$ with ϵ_V small enough (specified later).
2. Choose $\xi\in\mathbb{S}$ such that $W_1(\delta_\xi,\mu^{\mathrm{in}}) < \epsilon_V$. This is possible by the representation of the Wasserstein-1 distance (1a).
3. For any $\zeta > \epsilon_V$, we have
$$\int_{(\xi-\zeta,\xi+\zeta)} \mathrm{d}\mu^{\mathrm{in}} > 1 - \frac{\epsilon_V}{\zeta}. \tag{6}$$
4. Trace particles $\phi_1(t) := \Phi(t,\xi-\zeta,\mu^{\mathrm{in}})$ and $\phi_2(t) := \Phi(t,\xi+\zeta,\mu^{\mathrm{in}})$.
5. Because particles can never intersect (6) stays true for all $t\ge0$.
6. For small enough ζ, the phase difference $\Psi(t) := \phi_2(t)-\phi_1(t)$ satisfies
$$\dot\Psi < -\Psi\frac{g'(0)}{2}\left(1-\frac{\epsilon_V}{\zeta}\right) + 2\|f\|_\infty\frac{\epsilon_V}{\zeta}.$$
7. For small $\frac{\epsilon_V}{\zeta}$ we have $\Psi(t)\le\Psi(0)$ for all $t\ge0$.
8. The Wasserstein-1 distance can estimated from above by
$$W_1(\delta_{\phi_1(t)},\mu_t) \le \pi\frac{\epsilon_V}{\zeta} + 2\zeta.$$
9. Choose both ζ and $\frac{\epsilon_V}{\zeta}$ so small that $\pi\frac{\epsilon_V}{\zeta} + 2\zeta < \epsilon_U$.

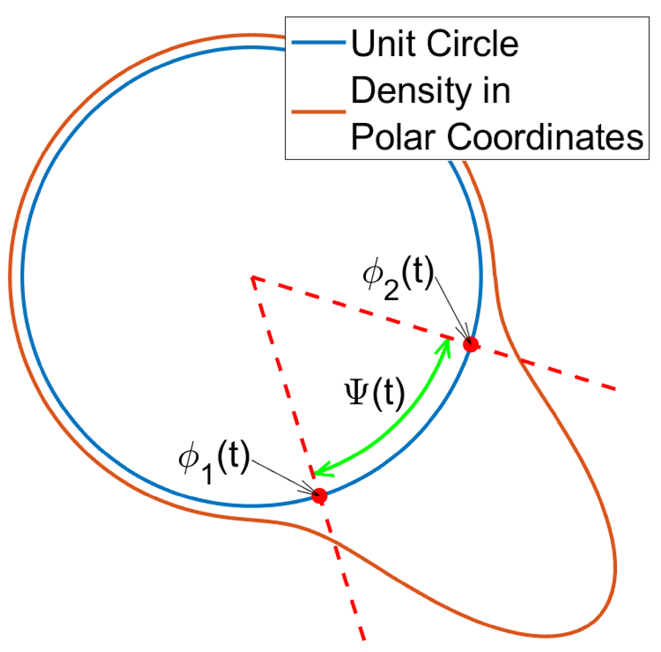

Fig. 1: Trapping most mass inside $(\phi_1(t),\phi_2(t))$.

Example

In [3], the author considered networks of $M=3$ finite coupled phase oscillator populations with higher-order interactions. Our results allows to analyze the stability of invariant sets of these networks in the mean-field limit. By choosing multi-indices $r^1=(3,1), r^2=(1,3), r^3=(2,1)$ and coupling functions $g_1(\alpha,\phi)=g_2(\alpha,\phi)=g_3(\alpha,\phi):=g(\alpha,\phi)$, with

$$g(\alpha,\gamma) = h_2(\gamma) - K^- h_4(\alpha_1,\gamma) + K^+ h_4(\alpha_2,\gamma),$$

and Lipschitz-continuous functions h_2, h_4, we can put the mean-field limit of the system from [3] into our framework. For example Theorem 2 yields the stability of the all-synchronized state if the function

$$f(\gamma) := g(0,\gamma) = h_2(\gamma) + (K^+-K^-)h_4(0,\gamma)$$

satisfies $f'(0)>0$.

References

[1] C Villani. *Topics in optimal transportation*. American Mathematical Soc., 2003.
[2] François Golse. Mean Field Kinetic Equations. *Course of Polytechnique*, 2013.
[3] Christian Bick. Heteroclinic Dynamics of Localized Frequency Synchrony: Heteroclinic Cycles for Small Populations. *Journal of Nonlinear Science*, 29(6):2547–2570, 2019.