Towards Abstract Wiener Model Spaces

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Abstract

Abstract Wiener spaces are in many ways the decisive setting for fundamental results on Gaussian measures: large deviations (Schilder), quasi-invariance (Cameron–Martin), differential calculus (Malliavin), support description (Stroock–Varadhan), concentration of measure (Fernique),... Analogues of these classical results have been derived in the "enhanced" context of Gaussian rough paths and, more recently, regularity structures equipped with Gaussian models. The aim of this article is to propose a notion of *abstract Wiener model space* that encompasses the aforementioned. More specifically, we focus here on enhanced Schilder type results, Cameron–Martin shifts and Fernique estimates, offering a somewhat unified view on results in [1] and [3].

Singular Stochastic PDEs

Consider a singular stochastic PDE, e.g. the stochastic Allen–Cahn equation

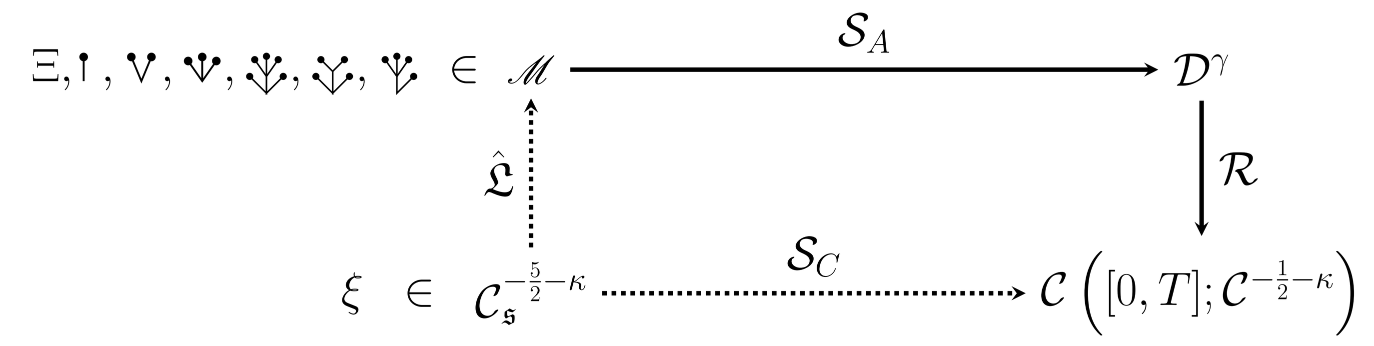
$$\underbrace{(\partial_t - \Delta)}_{\text{heat operator}}\phi = \underbrace{\phi - \phi^3}_{\text{non-linearity}} + \underbrace{\xi}_{\text{space-time white noise}}, \quad \phi(0,\cdot) = \phi_0, \quad \text{on } [0,T]\times\mathbb{T}^d. \tag{1}$$

Given a ϕ_0, a solution to (1) is a space-time function depending on the noise realization $\xi(\omega)$. That is, the solution map $\mathcal{S}_C$ is a non-linear operator on a probability space (E,μ), where E is a (separable Banach) space of realizations of ξ and μ is the distribution of ξ, which is a **Gaussian measure**.

Formally, a space-time white noise $\xi = \{\xi_z\}_{z\in[0,T]\times\mathbb{T}^3}$ is a family of random variables such that $\xi_z \perp \xi_{z'}$ if $z \neq z'$ and $\xi_z \sim \mathcal{N}(0,1)$. As a result, generic realizations of ξ are (very!) discontinuous. More precisely, $\xi \in \mathcal{C}_\mathfrak{s}^{-\frac{d+2}{2}-\kappa}([0,T]\times\mathbb{T}^d)$ for any $\kappa > 0$. Then via Schauder estimates $\phi \in \mathcal{C}_\mathfrak{s}^{-\frac{d+2}{2}-\kappa+2}([0,T]\times\mathbb{T}^d)$, which gives Hölder–Besov regularity $-\frac{5}{2}-\kappa+2 = -\frac{1}{2}-\kappa < 0$ for $d=3$. Thus, the cubic term ϕ^3 in (1) does not have canonical meaning; that is, the SPDE (1) is **singular** and not classically well-defined. However, (1) can be given meaning in the context of **regularity structures** introduced in [2].

Regularity Structures & Admissible Models

For $d=3$, due to the irregularity of ξ the classical solution map $\mathcal{S}_C$ below is *discontinuous*. However, it is discontinuous in a very specific way, namely through forming the non-linear terms in a Picard iteration of the right hand side of (1). The core idea of rough path theory and later regularity structures is to factor $\mathcal{S}_C$ through 1) a lift $\hat{\mathfrak{L}}$ assigning to a realization of ξ its non-linearities in the form of **rough paths/admissible models**, 2) an abstract solution map $\mathcal{S}_A$ from a space of models $\mathscr{M}$ to a space of abstract solutions $\mathcal{D}^\gamma$, and 3) a reconstruction operator $\mathcal{R}$ assigning to an abstract solution a concrete solution.



The space of models $\mathscr{M}$ is a non-linear subspace of a Banach space $\mathbf{E} = \oplus_\tau E_\tau$ which is graded by symbols $\tau \in \{\Xi, \dots\}$ associated to non-linearities. Due to the stronger topology on $\mathscr{M}$, the abstract solution map $\mathcal{S}_A$ is continuous, restoring well-posedness (in an appropriate sense). However, the lift $\hat{\mathfrak{L}}$ (of course) cannot be continuous, but is only measurable and typically only defined up to a.s. equivalence.

Abstract Wiener Model Space

Definition (Abstract Wiener Model Space)**.** An **abstract Wiener model space** (AWMS) is a quintuple $((\mathcal{T},\mathbf{E},[\cdot],\mathcal{N}),(\boldsymbol{\mathscr{H}},\iota),\boldsymbol{\mu},\mathfrak{L},\hat{\mathfrak{L}})$ consisting of

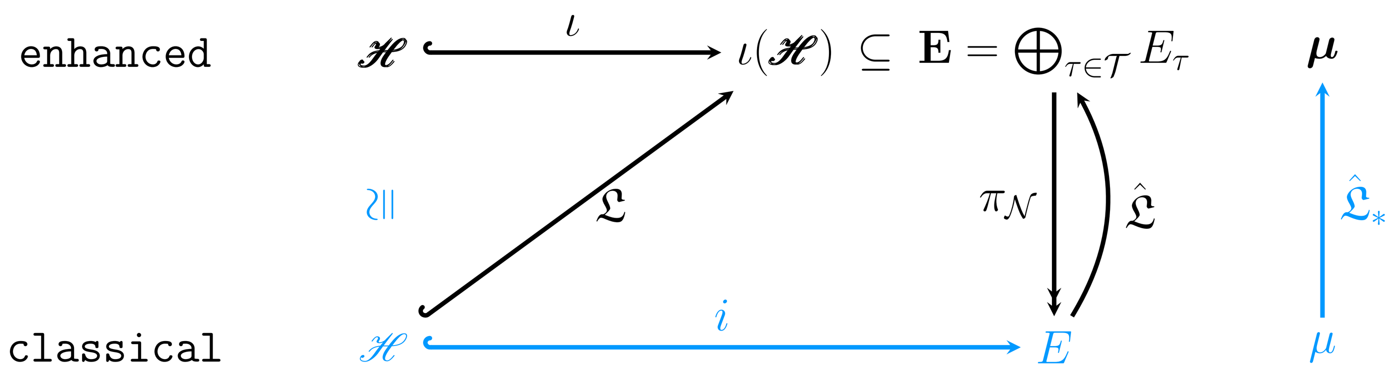
(1) a separable Banach space $\mathbf{E} = \bigoplus_{\tau\in\mathcal{T}} E_\tau$, graded over a finite set $\mathcal{T}$, together with a "degree" $[\cdot]: \mathcal{T}\to\mathbb{N}_{\geq 1}$ and a distinguished subset $\mathcal{N} \subseteq \{\tau\in\mathcal{T}\,|\,[\tau]=1\}$,

(2) a separable Hilbert space $\boldsymbol{\mathscr{H}}$ together with a continuous (in general non-linear) injection $\iota: \boldsymbol{\mathscr{H}} \hookrightarrow \mathbf{E}$, called **enhanced Cameron–Martin space**,

(3) a Borel probability measure $\boldsymbol{\mu}$ on $\mathbf{E}$, called **enhanced measure**, such that $\mu := (\pi_\mathcal{N})_*\boldsymbol{\mu}$ is centred Gaussian on E and $\mathscr{H} := \pi_\mathcal{N}(\iota(\boldsymbol{\mathscr{H}}))$ is the Cameron–Martin space associated to μ, where $\pi_\mathcal{N}: \mathbf{E} \to \oplus_{\tau\in\mathcal{N}} E_\tau$ denotes the canonical projection,

(4) a continuous lift $\mathfrak{L}: \mathscr{H} \to \mathbf{E}$ which is a left inverse of $\pi_\mathcal{N}|_{\iota(\boldsymbol{\mathscr{H}})}$, called **$\mathscr{H}$-skeleton lift**,

(5) a μ-a.s. equivalence class represented by measurable lifts $\hat{\mathfrak{L}}_\tau \in \mathcal{P}^{(\leq[\tau])}(E,\mu;E_\tau)$, called **full lift**, s.t. $\hat{\mathfrak{L}}_*\mu = \boldsymbol{\mu}$ and $\overline{\hat{\mathfrak{L}}_\tau} = \mathfrak{L}_\tau$, where $\overline{(\cdot)}$ denotes the proxy-restriction.



Gaussian Measures & Abstract Wiener Spaces

For a Gaussian measure μ on an infinite dimensional space a very interesting phenomenon occurs: the algebraic data (covariance form) and the analytic data (support) "detach" from one another. While the covariance form determines a separable Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle_\mathscr{H})$, the measure μ is not supported on $\mathscr{H}$, but on a *strictly larger* (Banach) space E in which $\mathscr{H}$ is contained via a linear injection i. In the case of space-time white noise this is given by

$$\underbrace{L^2([0,T]\times\mathbb{T}^d)}_{=\mathscr{H}} \overset{i}{\hookrightarrow} \underbrace{\mathcal{C}_\mathfrak{s}^{-\frac{d+2}{2}-\kappa}([0,T]\times\mathbb{T}^d)}_{=E}. \tag{2}$$

Such a quadruple $(E,\mathscr{H},i,\mu)$ is called **abstract Wiener space**.

Fundamental Theorems of Gaussian Measures

Let $(E,\mathscr{H},i,\mu)$ be an abstract Wiener space. Then the following hold:

Schilder's Large Deviation Principle: The family $(\mu(\varepsilon^{-1}(\cdot)))_{\varepsilon>0}$ satisfies a large deviation principle (LDP) on E with speed ε^2 and good rate function given by

$$\mathscr{I}(x) = \begin{cases} \frac{1}{2}\|x\|^2_\mathscr{H} & x\in\mathscr{H} \\ +\infty & \text{else.}\end{cases} \tag{3}$$

Cameron–Martin (CM) Theorem and Formula: For any $x\in E$ the measures $\mu(\cdot)$ and $\mu(\cdot - x)$ are equivalent if and only if $x\in\mathscr{H}$. Otherwise they are mutually singular.

Malliavin Calculus: The distribution of a (non-linear) Wiener functional $\Psi: E\to\mathbb{R}$ has a density with respect to the Lebesgue measure whenever the $\mathscr{H}$-derivative/Malliavin derivative (not the Fréchet derivative) of Ψ is non-degenerate.

Support Theorem: The topological support of μ in E equals the E-closure of $\mathscr{H}$.

Fernique Estimates: The random variable $x\mapsto\|x\|_E$ has Gaussian tails with decay rate controlled by the values of the $\mathscr{H}$-norm on the unit sphere in E.

Notice that all of these statements involve the properties of the Cameron–Martin space $(\mathscr{H},i)$ in a central way and those of E only in an auxiliary way. These classical theorems have their analogues in the "enhanced" framework of regularity structures. A natural question is therefore what an appropriate analogue of $(\mathscr{H},i)$ in that setting is.

Proxy-Restriction

Motivated by examples, one would like to simply *define* the inclusion of the "enhanced" Cameron–Martin space as $\iota(\boldsymbol{\mathscr{H}}) := \hat{\mathfrak{L}}(\mathscr{H})$. However, $\hat{\mathfrak{L}}$ is typically only defined up to μ-a.s. equivalence and $\mu(\mathscr{H}) = 0$ whenever $\dim(\mathscr{H}) = \infty$; i.e. "the restriction" of $\hat{\mathfrak{L}}$ to $\mathscr{H}$ is an ill-defined notion. As it turns out, due to the fact that the lift $\hat{\mathfrak{L}}_\tau$ associated to a symbol τ, lies in the $[\tau]$-th inhomogeneous E_τ-valued Wiener–Ito chaos (WIC) $\mathcal{P}^{(\leq[\tau])}(E,\mu;E_\tau)$ there is an appropriate definition of restriction (already considered in [3]):

$$\overline{\hat{\mathfrak{L}}_\tau}(h) := \mathbb{E}\left[\left(\Pi_{[\tau]}\hat{\mathfrak{L}}_\tau\right)\circ T_h\right], \quad h\in\mathscr{H}, \tag{4}$$

where $\Pi_{[\tau]}$ denotes the projection onto the $[\tau]$-th homogeneous WIC and $T_h(x) = x+h$ denotes the shift operator on E. In this context we refer to $\overline{\hat{\mathfrak{L}}_\tau}$ as the **proxy-restriction**.

Key Theorems

Let $((\mathcal{T},\mathbf{E},[\cdot],\mathcal{N}),(\boldsymbol{\mathscr{H}},\iota),\boldsymbol{\mu},\mathfrak{L},\hat{\mathfrak{L}})$ be an abstract Wiener model space.

Theorem 1 (Schilder LDP for AWMS)**.** *The sequence of naturally rescaled measures* $\left(\boldsymbol{\mu}\left(\sum_{\tau\in\mathcal{T}}\varepsilon^{-[\tau]}\pi_\tau(\cdot)\right)\right)_{\varepsilon>0}$ *satisfies a large deviation principle with good rate function*

$$\mathscr{I}(\mathbf{x}) = \begin{cases} \frac{1}{2}\|\pi_\mathcal{N}(\mathbf{x})\|^2_\mathscr{H} & \mathbf{x}\in\iota(\boldsymbol{\mathscr{H}}) \\ +\infty & \textit{else.}\end{cases} \tag{5}$$

Theorem 2 (Fernique Estimate for AWMS)**.** *There exists an* $\eta_0 > 0$ *such that*

$$\boldsymbol{\mu}(\mathbf{x}\in\mathbf{E}: \|\,\mathbf{x}\,\|_\mathbf{E} \geq t) \lesssim \exp\left(-\eta t^2\right), \quad \forall\eta>\eta_0, t\geq 0. \tag{6}$$

Theorem 3 (CM Theorem for AWMS)**.** *For any* $h\in\mathscr{H}$: $(\hat{\mathfrak{L}}\circ T_h)_*\mu \approx \boldsymbol{\mu}$ *and*

$$\frac{\mathrm{d}(\hat{\mathfrak{L}}\circ T_h)_*\mu}{\mathrm{d}\boldsymbol{\mu}}(\mathbf{x}) = \exp\left(\langle h, \pi_\mathcal{N}(\mathbf{x})\rangle - \frac{1}{2}\|h\|^2_\mathscr{H}\right). \tag{7}$$

Selected References

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