Inhomogeneous variants of the contact process

Eszter Couillard
Technical University of Munich

Abstract

The contact process was introduced by T. E. Harris in [3]. Since then, multiple variants have been studied.
Here: inhomogeneous variants of the contact process and other closely related processes. Inhomogeneous means that instead of parametrizing our process by a single infection rate, we look at models with multiple infection rates depending on the relative position of a node to the infected node. We will mainly concentrate on the long-range contact process and the inhomogeneous crabgrass model.
We give a sufficient condition under which the long-range process dies out a.s. and a sufficient condition under which the process in dimension at least 3 survives with positive probability.
The inhomogeneous crabgrass model is a generalization of the model introduced by M. Bramson, R. Durrett, and G. Swindle in [1]. We adapted some of their results for measures other than the uniform measure.

Generalize contact process

Let $\Lambda \in (0,\infty)$ and μ be a probability measure on $\mathbb{Z}^d$ such that:

$$\mu(\{0\}) = 0$$
$$\mu(\{x\}) = \mu(\{-x\}) \text{ for all } x \in \mathbb{Z}^d \quad (1)$$

Let $(\eta_t)_{t\geq 0}$ be a Feller-process that takes values in $\{0,1\}^{\mathbb{Z}^d}$
$(\eta_t)_{t\geq 0}$ is a generalized contact process with parameter Λ and μ if its generator is given by

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d}\left(\eta(x) + (1-\eta(x))\Lambda\sum_{y\in\mathbb{Z}^d}\eta(y)\mu(\{x-y\})\right)(f(\eta^x) - f(\eta))$$

for $f \in C(\{0,1\}^{\mathbb{Z}^d})$ and with

$$\eta^x(y) = \begin{cases} \eta(y) & \text{if } x \neq y \\ 1-\eta(y) & \text{if } x = y \end{cases} \quad \forall x,y \in \mathbb{Z}^d$$

Construction of the generalized contact process

Model a generalized contact process with parameter Λ and μ using the following independent Poisson point processes:

- $(N_x(t))_{t\geq 0}$ for $x \in \mathbb{Z}$ a Poisson point process with rate 1. Call them the cure points of x.
- $(N_{x,y}(t))_{t\geq 0}$ for $x,y \in \mathbb{Z}$ $x \neq y$ a Poisson point process with rate $\Lambda\mu(\{y-x\})$. Call them the infection arrows from x to y.

Given an initial configuration $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, the process evolves such that at any time point $t > 0$:

- If for some $x \in \mathbb{Z}^d$ $N_x(t) = 1$ then $\eta_t(x) = 1$.
- If for some $x,y \in \mathbb{Z}^d$ $N_{x,y}(t) = 1$ $x \neq y$ and $\eta_t(x) = 1$ then $\eta_t(y) = 1$.
- Otherwise, nothing changes.

The question

We say that the long-range contact process $(\eta_t)_{t\geq 0}$ **survives with positive probability** if

$$\mathbb{P}^{\{0\}}(\eta_t \neq \emptyset,\ \forall t \geq 0) > 0$$

Otherwise we say that$(\eta_t)_{t\geq 0}$ **dies-out a.s.**

Lemma 1
Let μ be a probability measure satisfying (1).
There exists $\Lambda_c(\mu) \in [0,\infty]$ such that for $\Lambda > 0$ and $(\eta_t)_{t\geq 0}$ a generalized contact process with parameters Λ and μ

$$\Lambda > \Lambda_c(\mu) \quad\Rightarrow\quad (\eta_t)_{t\geq 0} \textit{ survives with positive probability}$$
$$\Lambda < \Lambda_c(\mu) \quad\Rightarrow\quad (\eta_t)_{t\geq 0} \textit{ dies a.s.}$$

Question
Given a measure μ that satisfies (1), what is $\Lambda_c(\mu)$?

The critical value of the long-range contact processes

A generalized contact process with parameter Λ and μ is a long-range contact process if

$$\forall x,y \in \mathbb{Z}^d \quad ||x||_1 = ||y||_1 \Rightarrow \mu(\{x\}) = \mu(\{y\}) \quad (2)$$

Let μ be a probability measure on $\mathbb{Z}^d$ satisfying (2).
Then for all $d \geq 1$

$$\Lambda_c(\mu) \geq 1$$

For $d \geq 3$ and with γ_d the escape probability of the simple random walk in d dimensions:

$$\Lambda_c(\mu) \leq \frac{1}{2\gamma_d - 1}\sum_{k=1}^{\infty}\frac{\#\{y \in \mathbb{Z}^d : ||y||_1 = k\}}{2d}\mu(\{x : ||x||_1 = k\})$$

Key point in the proof: the link between the contact process and the escape probability of random walks shown by D. Griffeath in [2]

Rescaling the contact process

Let $(\mu_n)_{n\geq 1}$ be a sequence of measures such that μ_n is supported on the ball of radius n (with respect to some norm) satisfying (2) in that same norm.
Let $((\eta^n_t)_{t\geq 0})_{n\geq 1}$ a sequence of contact process with parameter $\Lambda > 0$ and $(\mu_n)_{n\geq 1}$.
The rescaled processes $(\zeta^n_t)_{t\geq 0}$ are taking values in $\{0,1\}^{\{k/n : k\in\mathbb{Z}^d\}}$. Rescale such that $\forall t \geq 0$ and $\forall x \in \mathbb{Z}^d$

$$\zeta^n_t(x/n) = \eta^n_t(x)$$

We say that rescaled processes are parametrized by $\Lambda > 0$ and the measure sequence $(\nu_n)_{n\geq 1}$ with ν_n the measure on $\mathbb{R}^d$ supported on $\{k/n : k \in \mathbb{Z}^d\}$ such that

$$\nu_n(A) = \mu_n(\{a \cdot n : a \in A\})$$

Crabgrass model

Let $((\zeta^n_t)_{t\geq 0})_{n\geq 1}$ a sequence of rescaled contact process parametrized by $\Lambda > 0$ and $(\nu_n)_{n\geq 1}$ with ν_n the uniform measure on $\{k/n : k \in \mathbb{Z}^d \setminus \{0\}$ with $||k/n||_\infty \leq 1\}$.

Theorem 2 (M. Bramson, R. Durrett, and G. Swindle in [1])
As $n \to \infty$, $\Lambda_c(\nu_n) \to 1$. Furthermore,

$$\Lambda_c(\nu_n) - 1 \approx \begin{cases} C/n^{2/3} & d = 1 \\ C\log(n)/n^2 & d = 2 \\ C/n^d & d \geq 3 \end{cases}$$

where $\approx$ means that if C is small then the right-hand side is a lower bound for large n and if C is big it is an upper bound.

Generalizing the crabgrass model

Let $((\zeta^n_t)_{t\geq 0})_{n\geq 1}$ a sequence of rescaled contact process parametrized by $\Lambda > 0$ and $(\nu_n)_{n\geq 1}$ measures supported on $\{k/n : k \in \mathbb{Z}^d \setminus \{0\}$ with $||k/n||_1 \leq 1\}$

Theorem 3 (Lower-bound)
Let

$$m(\nu_n) = \min(\{\nu_n(x) : x \in B_n(0) \text{ and } \nu_n(x) \neq 0\})$$

For n large enough (such that $m(\nu_n) \leq 0.36$) it holds

$$\Lambda_c(\nu_n) \geq 1 + \frac{1}{9}m(\nu_n)$$

Theorem 4 (Upper-bound)
Assume that the sequence $(\nu_n)_{n\geq 1}$ has a weak limit ν_∞.
Assuming that ν_∞ is atomless and that there is a $\rho > 0$ and $M_0 \in \mathbb{N}$ such that for all $n \geq M_0$ (or $n = \infty$) we have $\mathcal{Q}(\nu_n, 1) \leq 1 - \rho$ where $\mathcal{Q}(\cdot,\cdot)$ is the concentration function.
Then $\forall \varepsilon \in (0, 1/2)$ there exists an $M^\varepsilon_0 \in \mathbb{N}$ such that $\forall n \geq M^\varepsilon_0$

$$\Lambda_c(\nu_n) < 1 + \varepsilon$$

References

[1] M. Bramson, R. Durrett, and G. Swindle. "Statistical Mechanics of Crabgrass". In: *The Annals of Probability* 17.2 (1989), pp. 444–481.
[2] David Griffeath. "The Binary Contact Path Process". In: *The Annals of Probability* 11.3 (1983), pp. 692–705.
[3] T. E. Harris. "Contact Interactions on a Lattice". In: *The Annals of Probability* 2.6 (1974), pp. 969–988.