Tensor triangular geometry with some applications

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Abstract

Tensor triangular geometry provides an abstract framework in which classification problems in the context of tensor triangulated categories can be studied. One of the general goals of this theory can be described as the attempt to gather ideas and techniques from different areas of mathematics (e.g., algebraic geometry, modular representation theory) and unify them, so that one obtains statements for general tensor triangulated categories. This is particularly beneficial to develop methods that are only exploited thus far in specific areas. We discuss some core concepts of tensor triangular geometry in the sense of Balmer [1] with a particular emphasis on its connection to lattice theory and apply them in concrete settings.

Preliminaries I: Tensor triangulated categories

A tt-category (short for *tensor triangulated category*) is a triangulated category $(\mathcal{K}, \Sigma)$ equipped with a symmetric monoidal structure such that the symmetric monoidal product $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is a triangulated functor in each variable. In addition, for all objects $a \in \mathcal{K}$, the natural isomorphism ${}^a\theta = ({}^a\theta_b\colon a \otimes \Sigma(b) \cong \Sigma(a \otimes b))_{b\in\mathcal{K}}$ turning $a \otimes (-)\colon \mathcal{K} \to \mathcal{K}$ into a triangulated functor is required to be natural in $a \in \mathcal{K}$.
A tt-functor (short for *tensor triangulated functor*) is a functor between tt-categories that is both triangulated and (strong) symmetric monoidal.

Examples of essentially small tt-categories: (a) The category of perfect complexes: $(\mathrm{Perf}(X), \otimes^{\mathbf{L}}_{\mathcal{O}_X}, \mathcal{O}_X)$ for a qcqs (short for *quasi-compact quasi-separated*) scheme X. In particular, this also covers the affine case $(\mathrm{Perf}(R), \otimes^{\mathbf{L}}_R, R)$ for a commutative ring R.
(b) The stable module category: $(\mathrm{stmod}(kG), \otimes_k, k)$ for a field k and a finite group G.

In the sequel, $\mathcal{K} = (\mathcal{K}, \otimes, \mathbf{1})$ denotes an essentially small tt-category.

Preliminaries II: The Hochster dual

A subset Y of a spectral space X is a dual-open subset of X if it is of the form $Y = \bigcup_{i\in I} Y_i$ for closed subsets $Y_i \subseteq X$ with quasi-compact complement. The dual-open subsets of X define a topology on the underlying set of X (in terms of open subsets); the resulting topological space is again spectral and called the Hochster dual $X^\vee$ of X. The poset of dual-open subsets of X is denoted $\Omega(X^\vee) = (\Omega(X^\vee), \subseteq)$.

The heart of tt-geometry: The Balmer spectrum

A non-empty full subcategory $\mathcal{J}$ of $\mathcal{K}$ is called a tt-ideal of $\mathcal{K}$ (short for *thick $\otimes$-ideal*) if $\mathcal{J}$ is a thick subcategory of $\mathcal{K}$ satisfying $a \otimes j \in \mathcal{J}$ for all $a \in \mathcal{K}, j \in \mathcal{J}$. A prime tt-ideal of $\mathcal{K}$ is a proper tt-ideal $\mathcal{P}$ of $\mathcal{K}$ with the property

$$\forall a, b \in \mathcal{K}\colon\ (a \otimes b \in \mathcal{P} \Longrightarrow a \in \mathcal{P} \text{ or } b \in \mathcal{P}).$$

We topologize the set of prime tt-ideals of $\mathcal{K}$, denoted $\mathrm{Spc}(\mathcal{K})$, by declaring a subset of $\mathrm{Spc}(\mathcal{K})$ to be closed if (and only if) it is of the form

$$\mathrm{Z}(A) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid A \cap \mathcal{P} = \emptyset\} \qquad \text{for some family } A \subseteq \mathcal{K} \text{ of objects.}$$

If $A = \{a\}$ for $a \in \mathcal{K}$, we write $\mathrm{supp}(a) := \mathrm{Z}(A)$ for the support of a. The topological space $\mathrm{Spc}(\mathcal{K})$ is referred to as the Balmer spectrum of $\mathcal{K}$.
A tt-ideal $\mathcal{J}$ of $\mathcal{K}$ is a radical tt-ideal if it contains an object $a \in \mathcal{K}$, whenever there exists an integer $n \geq 1$ such that $a^{\otimes n} = a \otimes \ldots \otimes a \in \mathcal{J}$. The set of all radical tt-ideals of $\mathcal{K}$ is denoted $\mathrm{Zar}(\mathcal{K})$ and called the Zariski frame of $\mathcal{K}$; it is ordered by inclusion.

Remark: Since $\mathcal{K}$ is essentially small, the collection of its tt-ideals forms a *set*. In particular, $\mathrm{Spc}(\mathcal{K})$ and $\mathrm{Zar}(\mathcal{K})$ are indeed both sets.

Theorem ([1], [2]): (a) The Zariski frame $\mathrm{Zar}(\mathcal{K})$ is a coherent frame.
(b) The space $\mathrm{Spc}(\mathcal{K})$ is equal to the Hochster dual of the spectral space of $\wedge$-prime elements of $\mathrm{Zar}(\mathcal{K})$. In particular, the Balmer spectrum is a spectral space.
(c) Assigning to a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ the spectral map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$, $\mathcal{Q} \mapsto F^{-1}(\mathcal{Q})$ defines a contravariant functor from the category of essentially small tt-categories to the category of spectral spaces.

A first example

Let R be a Dedekind ring. For any prime ideal $\mathfrak{p}$ of R, define the tt-ideals

$$\mathcal{P}_\mathfrak{p} := \mathrm{thick}\big(R/\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}(R) \setminus \{0, \mathfrak{p}\}\big) \subseteq \mathrm{Perf}(R)$$

of $\mathrm{Perf}(R)$; note, by definition, $\mathrm{thick}(\emptyset) = 0$. Heavily exploiting properties characteristic of Dedekind rings (e.g., the structure theorem of finitely generated modules over a Dedekind ring), we prove that the assignment

$$\mathrm{Spec}(R) \longrightarrow \mathrm{Spc}\big(\mathrm{Perf}(R)\big), \ \ \mathfrak{p} \longmapsto \mathcal{P}_\mathfrak{p}$$

is a well-defined homeomorphism.

(Classifying) support data

A support datum on $\mathcal{K}$ is a pair (X, σ) consisting of a topological space X and an assignment σ that associates to each object a of $\mathcal{K}$ a closed subset $\sigma(a)$ of X subject to the following axioms:

- (SD1) $\sigma(0) = \emptyset$ and $\sigma(\mathbf{1}) = X$.
- (SD2) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$.
- (SD3) $\sigma\big(\Sigma(a)\big) = \sigma(a)$.
- (SD4) If $a \to b \to c \to \Sigma(a)$ is a distinguished triangle of $\mathcal{K}$, then $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$.
- (SD5) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$.

A morphism $f\colon (X, \sigma) \to (Y, \tau)$ of support data (on $\mathcal{K}$) is a continuous map $f\colon X \to Y$ such that $\sigma(a) = f^{-1}(\tau(a))$ for all $a \in \mathcal{K}$. This gives a category of support data on $\mathcal{K}$, denoted by $\mathrm{Support}(\mathcal{K})$.

Theorem ([1], [3]): (a) $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is a terminal object of $\mathrm{Support}(\mathcal{K})$.
(b) The inclusion-preserving map $\mathrm{Zar}(\mathcal{K}) \to \Omega(\mathrm{Spc}(\mathcal{K})^\vee), \mathcal{J} \mapsto \bigcup_{j\in\mathcal{J}} \mathrm{supp}(j)$ is an order isomorphism with inverse $Y \mapsto \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subseteq Y\}$.
(c) The terminal objects of $\mathrm{Support}(\mathcal{K})$ are precisely the classifying support data, that is, support data (X, σ) that have the following properties:

- (CSD1) X is a spectral space.
- (CSD2) For all $a \in \mathcal{K}$, the open subset $X \setminus \sigma(a)$ is a quasi-compact space.
- (CSD3) The map $\mathrm{Zar}(\mathcal{K}) \to \Omega(X^\vee), \mathcal{J} \mapsto \bigcup_{j\in\mathcal{J}} \sigma(j)$ is an order isomorphism.

Examples

Example (A): The Balmer spectrum of $\mathrm{Perf}(X)$ for a qcqs scheme X

Theorem: Together with the cohomological support

$$\mathrm{supph}(\mathcal{F}^\bullet) := \big\{x \in X \ \big|\ \mathcal{F}^\bullet_x \text{ is not exact}\big\} \subseteq X \qquad \text{for } \mathcal{F}^\bullet \in \mathrm{Perf}(X),$$

the underlying space $|X|$ of the given scheme X forms a classifying support datum on $\mathrm{Perf}(X)$. The assignment

$$\Big(|X|, \mathrm{supph}\Big) \cong \Big(\mathrm{Spc}\big(\mathrm{Perf(X)}\big), \mathrm{supp}\Big), \ \ x \longmapsto \Big\{\mathcal{F}^\bullet \in \mathrm{Perf}(X) \ \Big|\ \mathcal{F}^\bullet_x \text{ is exact}\Big\}$$

is the unique isomorphism of support data.
$\rightsquigarrow$ The proof closely follows [2] and uses a reduction to the affine case, Bousfield localization techniques as well as a deep result from algebraic K-theory.

Example (B): The Balmer spectrum of $\mathrm{stmod}(kG)$ for an algebraically closed field k of positive characteristic $p > 0$ and a finite group G whose order is divisible by p

Define the finitely generated commutative k-algebra $H^\bullet(G, k)$ by setting it to $H^*(G, k)$ if $p = 2$ and $\bigoplus_{i\in 2\mathbb{Z}_{\geq 0}} H^i(G, k)$ if p is odd. Given a kG-module M, denote $I_G(M)$ the kernel of the ring map $(-) \otimes_k M\colon H^\bullet(G, k) \to \mathrm{Ext}^*_{kG}(M, M)$.

Theorem: Together with the support

$$\mathcal{V}_G(M) := V_+\big(I_G(M)\big) \subseteq \mathrm{Proj}\big(H^\bullet(G, k)\big) \qquad \text{for } M \in \mathrm{stmod}(kG),$$

the projective spectrum $\mathrm{Proj}(H^\bullet(G, k))$ forms a classifying support datum on $\mathrm{stmod}(kG)$. The assignment

$$\Big(\mathrm{Proj}\big(H^\bullet(G, k)\big), \mathcal{V}_G\Big) \cong \Big(\mathrm{Spc}\big(\mathrm{stmod}(kG)\big), \mathrm{supp}\Big)$$
$$\mathfrak{p} \longmapsto \Big\{M \in \mathrm{stmod}(kG) \ \Big|\ I_G(M) \nsubseteq \mathfrak{p}\Big\}$$

is the unique isomorphism of support data.
$\rightsquigarrow$ The proof is based on results from [4] and an extension of $\mathcal{V}_G$ to arbitrary kG-modules.

The structure sheaf

The Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ naturally refines to a locally ringed space

$$\mathrm{Spec}(\mathcal{K}) := \big(\mathrm{Spc}(\mathcal{K}), \mathcal{O}_\mathcal{K}\big).$$

Furthermore, given a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$, the continuous map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ can be promoted to a morphism of locally ringed spaces.

Theorem ([1], [3]): The isomorphisms from Example (A) and (B) refine to isomorphisms of locally ringed spaces, that is, $X \cong \mathrm{Spec}(\mathrm{Perf}(X))$ and $\mathrm{Proj}(H^\bullet(G, k)) \cong \mathrm{Spec}(\mathrm{stmod}(kG))$.

Selected references

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[4] D. J. Benson, J. F. Carlson and J. Rickard. "Thick subcategories of the stable module category". In: *Fundamenta Mathematicae* **153** (1) (1997), pp. 59-80.