On Fundamental Functionals of Convex Geometry: New, Refined, and Corrected Results

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Abstract

One of the most basic, yet most important problems in computational geometry is determining the *circumradius* of a point set $P \subset \mathbb{R}^n$ with respect to some container $C \subset \mathbb{R}^n$, i.e. computing $R(P,C) \coloneqq \inf\{\lambda \geq 0 : P \subset \lambda C + t, t \in \mathbb{R}^n\}$. This problem and its variants give rise to numerous functionals, which form a central topic of study in convex geometry. We analyze the properties of some of these functionals over the set of *convex bodies* $\mathcal{C}^n$, i.e. non-empty compact convex subsets of $\mathbb{R}^n$ (or $\mathcal{C}^n_0$ for those with the origin in their interior). In particular, we extend results previously only obtained for *symmetric* bodies, i.e. bodies $C \in \mathcal{C}^n$ with $-(C-c) = (C-c)$ for some $c \in \mathbb{R}^n$, to general bodies $C \in \mathcal{C}^n$ and establish connections to its *Minkowski asymmetry* $s(C) \coloneqq R(-C,C)$ and *Minkowski centeredness*, i.e. when $-C \subset s(C)C$.

Support and Gauge Functions

The *support* and *gauge function* of a convex body $C \in \mathcal{C}^n_0$ are defined in order by

$$h_C \colon \mathbb{R}^n \to [0,\infty),\ h_C(a) \coloneqq \max\{a^T x : x \in C\},$$
$$\|\cdot\|_C : \mathbb{R}^n \to [0,\infty),\ \|x\|_C \coloneqq \min\{\lambda \geq 0 : x \in \lambda C\}.$$

Despite their simplicity, these functions are fundamental tools in many branches of convex geometry. For example, they encode the important separation theorems in the sense that two bodies are equal if and only if their support (or gauge) functions coincide. By analyzing the (local) Lipschitz continuity of $\|\cdot\|_C$ and $0 \neq x \mapsto \frac{x}{\|x\|_C}$, we could substantially weaken the assumptions in this fundamental property:

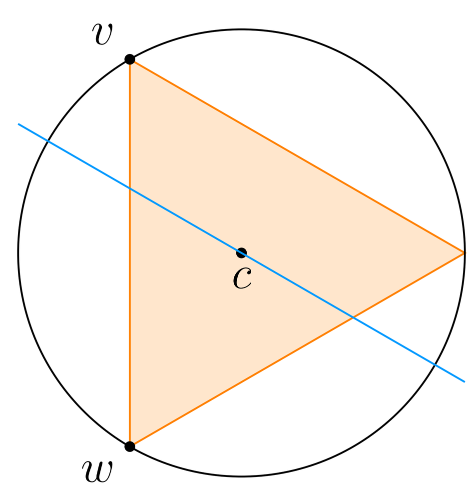
Theorem 1. *The gradients of the support (or gauge) functions of convex bodies $K, L \in \mathcal{C}^n_0$ are parallel at all points for which both gradients exist if and only if $n = 1$ or K and L are equal up to dilatation.*

If in addition to the above for $n \geq 2$ the support (or gauge) functions of K and L coincide at at least one non-zero point, then K and L must already be equal.

Halfspace Lemma

The so-called *Halfspace Lemma* states the following:

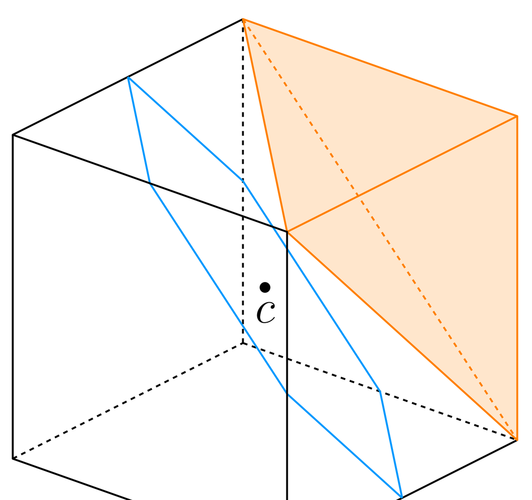
> *A Euclidean ball $C \subset \mathbb{R}^n$ contains a body $K \subset C$ optimally (i.e. $K \not\subset \lambda C + t$ for any $\lambda \in [0,1)$, $t \in \mathbb{R}^n$) if and only if every closed halfspace with the ball's center c in its boundary contains a common relative boundary point of K and C.*



An example of a triangle (orange) optimally contained in a circle. The Halfspace Lemma asserts that both closed halfspaces bounded by the blue line must contain a common relative boundary point of the triangle and the circle, here for example v and w.

This property is the basis for numerous results about Eudlidean radii, both of theoretical and practical nature. Therefore, it is of interest for which other choices of the convex body C and the point $c \in C$ this property remains valid. A result by Klee [3] shows that it, in fact, characterizes ellipsoids among c-symmetric bodies of dimension at least 3:

Proposition 2. *Let $C \in \mathcal{C}^n$ be c-symmetric for some $c \in \mathbb{R}^n$. Then the Halfspace Lemma is valid for the pair (C,c) if and only if $\dim(C) \in \{1,2\}$ and C is strictly convex (i.e. the relative boundary of C contains no segments), or $\dim(C) \geq 3$ and C is an ellipsoid.*



An example of a simplex (orange) optimally contained in a cube. The halfspace bounded by the plane which intersects the cube in the blue hexagon but does not contain the simplex contains no common (relative boundary) points of the simplex and the cube.

Using Theorem 1 (applied to $K = C - c$ and $L = c - C$), we could extend this characterization to all possible choices of C and c, without any symmetry assumptions:

Theorem 3. *Let $C \in \mathcal{C}^n$ and $c \in \mathbb{R}^n$. Then the Halfspace Lemma restricted to segments $K \subset C$ is valid for the pair (C,c) if and only if $\dim(C) = 1$ and $c \in \operatorname{relint}(C)$, or $\dim(C) \geq 2$ and C is c-symmetric and strictly convex.*

Corollary 4. *Let $C \in \mathcal{C}^n$ and $c \in \mathbb{R}^n$. Then the Halfspace Lemma is valid for the pair (C,c) if and only if $\dim(C) = 1$ and $c \in \operatorname{relint}(C)$, or $\dim(C) = 2$ and C is c-symmetric and strictly convex, or $\dim(C) \geq 3$ and C is a c-symmetric ellipsoid.*

Behavior of the Minkowski Asymmetry Under Polarization

The *polar* of a convex body $C \in \mathcal{C}^n_0$ is defined by

$$C^\circ \coloneqq \{a \in \mathbb{R}^n : a^T x \leq 1 \text{ for all } x \in C\}.$$

It can be understood as a generalization of the unit ball of a dual space for not necessarily symmetric gauges instead of norms, as well as of the inversion of numbers to convex bodies. Much like dual spaces in functional analysis, polar bodies arise as useful tools in certain areas of convex geometry. Despite the Minkowski asymmetry also arising naturally in many of the same settings, surprisingly no connections between them have been investigated so far. We begin to fill this gap with the following result:

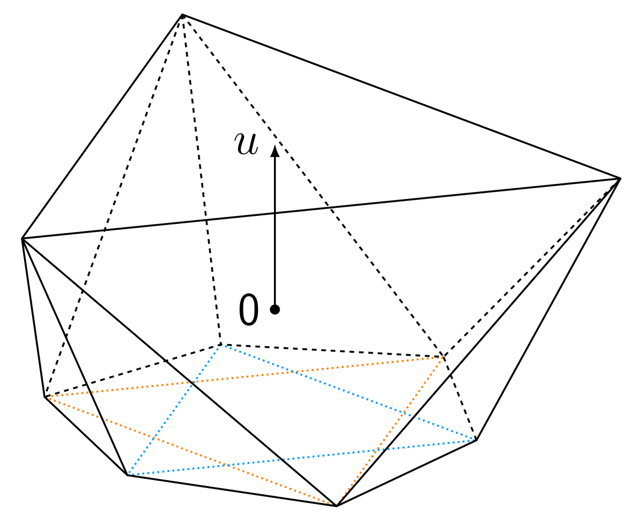
Lemma 5. *For a Minkowski centered body $C \neq \{0\}$, it holds*

$$s(C^\circ) \leq s(C),$$

with equality if and only if the Halfspace Lemma restricted to $K = -\frac{1}{s(C)}C$ is valid for the pair $(C,0)$, or equivalently if C° is Minkowski centered.

This leaves open whether the equality case applies, which is answered by the following:

Theorem 6. *If $C \in \mathcal{C}^n$ is Minkowski centered with $s(C) > n-1$, then $s(C^\circ) = s(C)$. In contrast, for any $s \in [1, n-1]$ and $t \in [1,s]$, there exists a Minkowski centered body $C \in \mathcal{C}^n$ with $s(C) = s$ and $s(C^\circ) = t$.*



The body C for $1 \leq t \leq s \leq n-1$ can be constructed as follows: Choose some $u \in \mathbb{R}^n \setminus \{0\}$ and a Minkowski centered $(n-1)$-simplex $K \subset H_{(u,0)}$. Then define C as the convex hull of the union of $K + u$ (black, top), $-\frac{1}{s}(K+u)$ (orange, bottom), and $-\frac{1}{s}((1 - t\frac{n}{n-1})K + u)$ (blue, bottom).

Core-, Cylinder-, and Intersection-Radii

The k-th *core-*, *cylinder-*, and *intersection-radius* of $K \in \mathcal{C}^n$ with respect to $C \in \mathcal{C}^n_0$ are for $k \in \{1, ..., n\}$ defined in order by

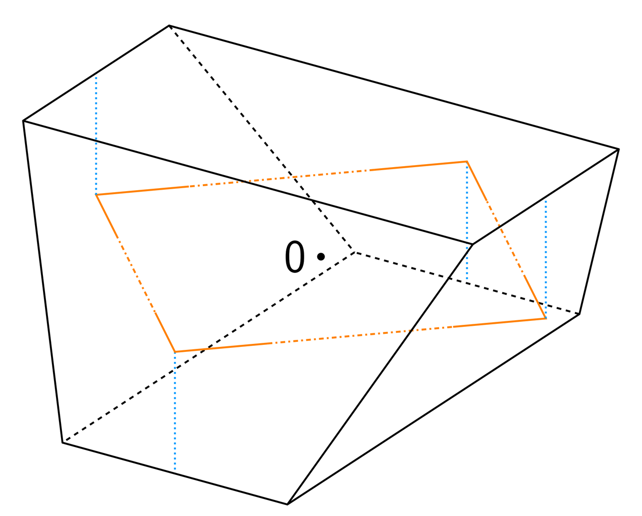
$$R_k(K,C) \coloneqq \max\{R(S,C) : S \subset K, |S| \leq k+1\},$$
$$R^\pi_k(K,C) \coloneqq \max\{R(K, C + L^\perp) : L \text{ is a linear subspace with } \dim(L) \leq k\},$$
$$R^\sigma_k(K,C) \coloneqq \max\{R(K \cap A, C) : A \text{ is an affine subspace with } \dim(A) \leq k\}.$$

These different types of radii are sometimes studied for the analysis of or use in (approximation) algorithms for problems e.g. in computer graphics or pattern recognition. There has been a misconception in the literature [1] that all three radii always coincide, which is unfortunately not true in general. While the first two radii are indeed always equal, the third may in certain configurations be larger.

Theorem 7. *Let $C \in \mathcal{C}^n_0$, $K \in \mathcal{C}^n$, and $k \in \{1, ..., n\}$. Then*

$$R_k(K,C) = R^\pi_k(K,C) \leq R^\sigma_k(K,C) \leq \min\left\{\frac{n}{k}, \frac{(s(C)+1)s(K)}{s(K)+1}\right\} R_k(K,C).$$

$R^\pi_k(K,C) = R^\sigma_k(K,C)$ is guaranteed if $k \in \{1,n\}$ or if C is an ellipsoid or a parallelotope, but can in general fail.



The parts of the orange square's relative boundary inside the body C are dash-dotted. The square's vertices are directly above/below midpoints of some edges of C (indicated by blue dotted lines). It can be shown that $R_2(K,C) = 1 < \frac{4}{3} = R^\sigma_2(K,C)$.

Selected References

[1] R. Brandenberg, S. König, *No Dimension-Independent Core-Sets for Containment Under Homothetics.* Discrete Comput. Geom., Vol. 49, No. 1 (2013), 3–21.

[2] V. L. Klee, *The Critical Set of a Convex Body.* Amer. J. Math., Vol. 75, No. 1 (1953), 178–188.

[3] V. L. Klee, *Circumspheres and Inner Products.* Math. Scand., Vol. 8, No. 2 (1960), 363–370.