Support and Gauge Functions

The support and gauge function of a convex body \( C \) are defined in order by

\[
\begin{align*}
    &h_C : \mathbb{R}^n \to [0, \infty), h_C(\alpha) = \max \{ \langle x, \alpha \rangle : x \in C \}, \\
    &\| |_C : \mathbb{R}^n \to [0, \infty), \| |_C(x) = \min \{ \lambda \geq 0 : x \in \lambda C \}.
\end{align*}
\]

Despite their simplicity, these functions are fundamental tools in many branches of convex geometry. For example, they encode the important separation theorems in the sense that two bodies are equal if and only if their support (or gauge) functions coincide. By analyzing the (local) Lipschitz continuity of \( \| |_C \) and \( h_C \), we could substantially weaken the assumptions in this fundamental property.

**Theorem 1.** The gradients of the support (or gauge) functions of convex bodies \( K, L \in C_0 \) are parallel at all points for which both gradients exist if and only if \( C \) is a parallelotope.

This leaves open whether the equality case applies, which is answered by the following:

**Lemma 5.** For a Minkowski centered body \( C \neq \emptyset \), it holds \( s(C) \leq s(C) \), with equality if and only if the Halfspace Lemma restricted to \( K = -\frac{1}{\lambda} C \) is valid for the pair \((\lambda C, 0)\) or equivalently \( C \) is Minkowski centered.

Halfspace Lemma

The so-called Halfspace Lemma states the following:

A Euclidean ball \( B \subset \mathbb{R}^n \) contains a body \( K \subset C \) optimally (i.e. \( K \not\subset C \) for any \( \lambda \in [0, 1], \lambda \in \mathbb{R}^n \)) if and only if every closed halfspace with the ball’s center \( c \) in its boundary contains a common relative boundary point of \( K \) and \( C \).

An example of a triangle (orange) optimally contained in a circle. The Halfspace Lemma asserts that both closed halfspaces bounded by the blue line must contain a common relative boundary point of the triangle and the circle, here for example \( v \) and \( w \).

This property is the basis for numerous results about Euclidean radii, both of theoretical and practical nature. Therefore, it is of interest for which other choices of the convex body \( C \) and the point \( c \in C \) this property remains valid. A result by Klee [3] shows that it, in fact, characterizes ellipsoids among \( c \)-symmetric bodies of dimension at least 3:

**Proposition 2.** Let \( C \in C_0 \) be \( c \)-symmetric for some \( c \in \mathbb{R}^n \). Then the Halfspace Lemma is valid for the pair \((c, c)\) if and only if \( \dim(C) \in \{1, 2\} \) and \( C \) is strictly convex (i.e. the relative boundary of \( C \) contains no segments), or \( \dim(C) \geq 3 \) and \( C \) is an ellipsoid.

An example of a simplex (orange) optimally contained in a cube. The halfspace bounded by the blue hexagon but does not contain the simplex contains no common (relative boundary) points of the simplex and the cube.

Using Theorem 1 (applied to \( K = C - c \) and \( L = C - c \)), we could extend this characterization to all possible choices of \( C \) and \( c \), without any symmetry assumptions:

**Theorem 3.** Let \( C \in C_0 \) and \( c \in \mathbb{R}^n \). Then the Halfspace Lemma restricted to segments \( K \subset C \) is valid for the pair \((c, c)\) if and only if \( \dim(C) = 1 \) and \( c \in \text{relint}(C) \), or \( \dim(C) \geq 2 \) and \( C \) is \( c \)-symmetric and strictly convex.

**Corollary 4.** Let \( C \in C_0 \) and \( c \in \mathbb{R}^n \). Then the Halfspace Lemma is valid for the pair \((c, c)\) if and only if \( \dim(C) = 1 \) and \( c \in \text{relint}(C) \), or \( \dim(C) = 2 \) and \( C \) is \( c \)-symmetric and strictly convex, or \( \dim(C) \geq 3 \) and \( C \) is a \( c \)-symmetric ellipsoid.

Behavior of the Minkowski Asymmetry Under Polarization

The polar of a convex body \( C \in C_0 \) is defined by

\[ C^\circ = \{ a \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \text{ for all } x \in C \}. \]

It can be understood as a generalization of the unit ball of a dual space for not necessarily symmetric gauges instead of norms, as well as of the inversion of numbers to convex bodies. Much like dual spaces in functional analysis, polar bodies arise as useful tools in certain areas of convex geometry. Despite the Minkowski asymmetry also arising naturally in many of the same settings, surprisingly no connections between them have been investigated so far. We begin to fill this gap with the following result:

**Lemma 6.** If \( C \in C_0 \) is Minkowski centered with \( s(C) > n - 1 \), then \( s(C^\circ) = s(C) \). In contrast, for any \( s \in [1, n-1] \) and \( t \in [1, s] \), there exists a Minkowski centered body \( C \in C_0^s \) with \( s(C) = s \) and \( s(C^\circ) = t \).

Selected References

