Phase Diagram of the Quantum Random Energy Model

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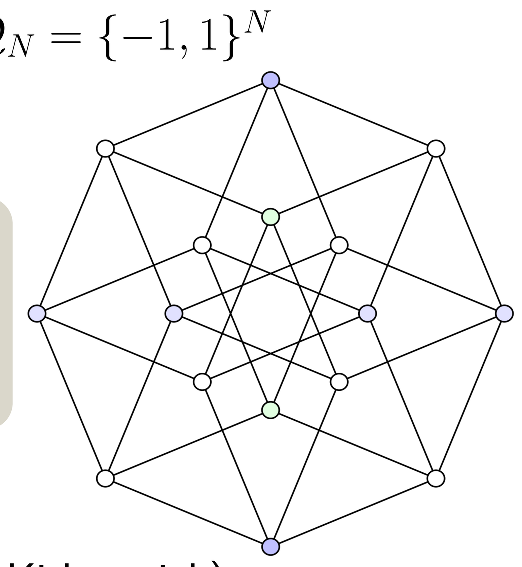
Abstract

The quantum random energy model (QREM) serves as a simple cornerstone and a testing ground, for a number of fields. It is the simplest of all mean-field spin glass models in which quantum effects due to the presence of a transversal field are studied. Renewed interest in its spectral properties arose recently in connection with quantum annealing algorithms [1,2] and many-body localisation [4,6]. In our paper [5] we prove Goldschmidt's formula [3] for the QREM's free energy. In particular, we verify the location of the first order and the freezing transition in the phase diagram. The proof avoids replica methods and is based on a combination of variational methods on the one hand, and percolation bounds on large-deviation configurations in combination with simple spectral bounds on the hypercube's adjacency matrix on the other hand.

The Quantum Random Energy Model

Configuration space of N spin-$\frac{1}{2}$ particles: Hamming cube $\mathcal{Q}_N = \{-1,1\}^N$



Random Energy Model (Derrida '80)

$$U(\sigma) := \sqrt{N} g(\sigma),$$

with $(g(\sigma))_\sigma$ i.i.d. process with standard normal law

REM is $p \to \infty$-limit of p-spin models ($p = 2$: Sherrington-Kirkpatrick)

$$\mathbb{E}[U(\sigma)] = 0, \quad \mathbb{E}[U(\sigma)U(\sigma')] = N \left(\frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma_i' \right)^p$$

Transversal magnetic field taken into account via

$$(T\psi)(\sigma) := -\sum_{j=1}^{N} \psi(\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_N), \quad \psi \in \ell^2(\mathcal{Q}_N) \simeq \bigotimes_{k=1}^{N} \mathbb{C}^2.$$

Quantum Random Energy Model

$$H := U + \Gamma T,$$

$\Gamma \geq 0$ strength of magnetic field.

Model for studying quantum effects, e.g. in mean-field spin glasses and quantum annealing algorithms, and mutation of genotypes in random fitness landscape.

Main Result

Partition function at inverse temperature $\beta \in [0, \infty)$: $Z(\beta, \Gamma) = 2^{-N} \mathrm{Tr} e^{-\beta H}$

Pressure: $p_N(\beta, \Gamma) := N^{-1} \ln Z(\beta, \Gamma)$

Thermodynamic limit ($N \to \infty$): the pressure of the REM converges almost surely

$$\lim_{N\to\infty} p_N(\beta, 0) = p^{\mathrm{REM}}(\beta) = \begin{cases} \frac{1}{2}\beta^2 & \text{if } \beta \leq \beta_c, \\ \frac{1}{2}\beta_c^2 + (\beta - \beta_c)\beta_c & \text{if } \beta \leq \beta_c. \end{cases}$$

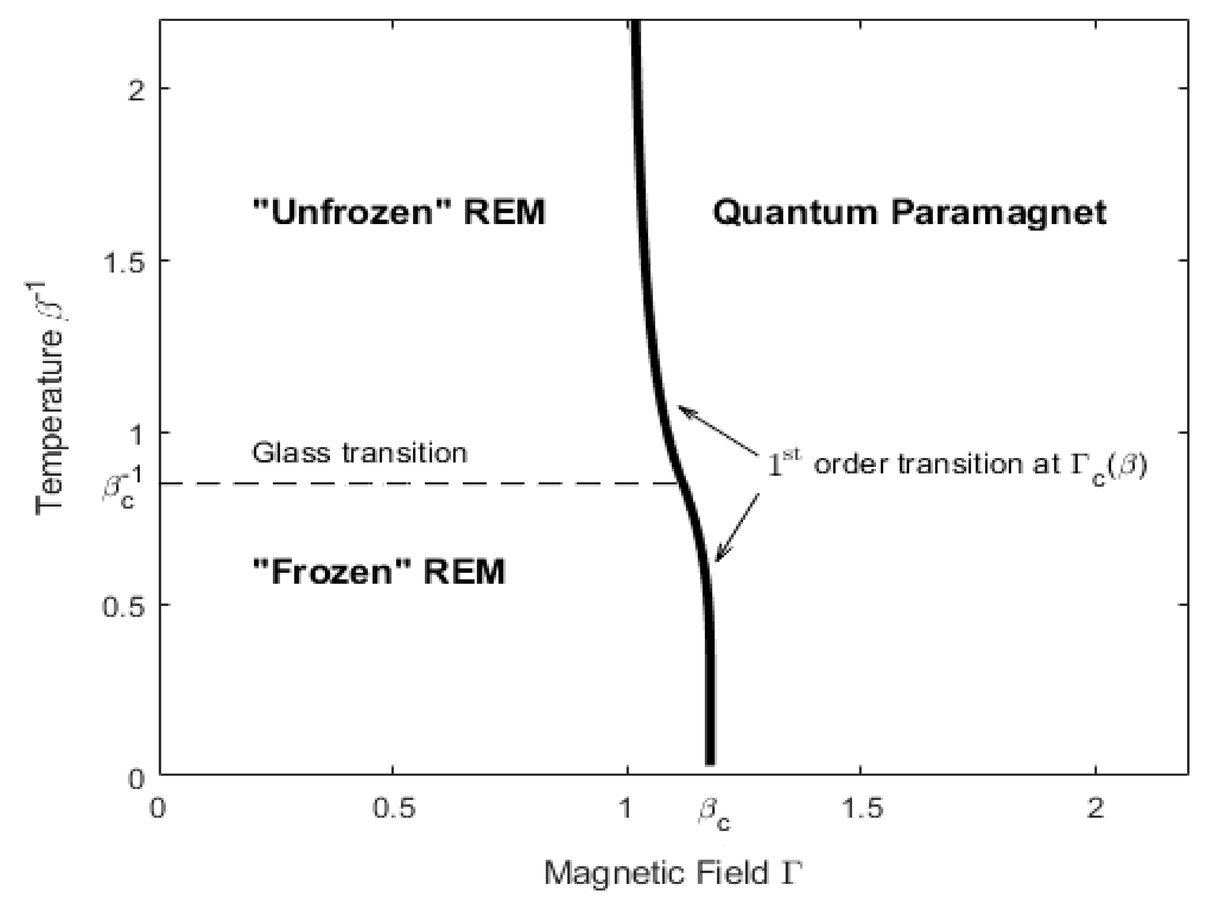
Freezing transition at inverse temperature $\beta_c = \sqrt{2 \ln 2}$, β_c coincides with specific ground state energy.

Paramagnetic pressure ($U = 0$): $\quad p^{\mathrm{PAR}}(\beta\Gamma) = \ln \cosh(\beta\Gamma)$

Theorem M./W. '19 For any $\Gamma, \beta \geq 0$, we have the almost sure convergence

$$\lim_{N\to\infty} p_N(\beta, \Gamma) = \max\{p^{\mathrm{REM}}(\beta), p^{\mathrm{PAR}}(\beta\Gamma)\}$$

Goldschmidt calculated the limit of the pressure via the (non-rigorous) replica method and static approximation in path-integral representation of $\mathbb{E}[Z(\beta, \Gamma)^n]$.



First-order phase transition found at $\Gamma_c(\beta) = \beta^{-}1 \mathrm{arcosh}(\exp(p^{\mathrm{REM}}(\beta)))$

$\Gamma < \Gamma_c(\beta)$: freezing transition unchanged at $\beta = \beta_c$

$\Gamma > \Gamma_c(\beta)$: magnetization in the x-direction equals $\tanh(\beta\Gamma) > 0$

Sketch of the Proof

Basic idea: prove pair of asymptotically coinciding upper and lower bound for $p_N(\beta, \Gamma)$.

1. Lower bound: Based on Gibbs variational principle

$$\ln \mathrm{Tr} e^{-\beta H} = - \inf_{\rho \text{ density matrix}} [\beta \mathrm{Tr}(H\rho) + \mathrm{Tr}(\rho \ln \rho)].$$

Pick REM Gibbs state $\rho = e^{-\beta U}/\mathrm{Tr} e^{-\beta U}$ and paramagnetic Gibbs state $\rho = e^{-\beta\Gamma T}/\mathrm{Tr} e^{-\beta\Gamma T}$:

$$p_N(\beta, \Gamma) - p^{\mathrm{REM}}(\beta) \geq -\frac{\beta}{N} \frac{\mathrm{Tr} T e^{-\beta U}}{\mathrm{Tr} e^{-\beta U}} = 0$$

$$p_N(\beta, \Gamma) - p^{\mathrm{PAR}}(\beta\Gamma) \geq -\frac{\beta}{N} \frac{\mathrm{Tr} U e^{-\beta\Gamma T}}{\mathrm{Tr} e^{-\beta\Gamma T}} = -\frac{\beta}{2^N \sqrt{N}} \sum_\sigma g(\sigma) = \mathcal{O}\left(\frac{1}{\sqrt{N 2^N}}\right)$$

$$\Rightarrow \liminf_{N\to\infty} p_N(\beta, \Gamma) \geq \max\{p^{\mathrm{REM}}(\beta), p^{\mathrm{PAR}}(\beta\Gamma)\} \text{ almost surely}$$

2. Upper bound: Consider for $\varepsilon > 0$ large deviation set

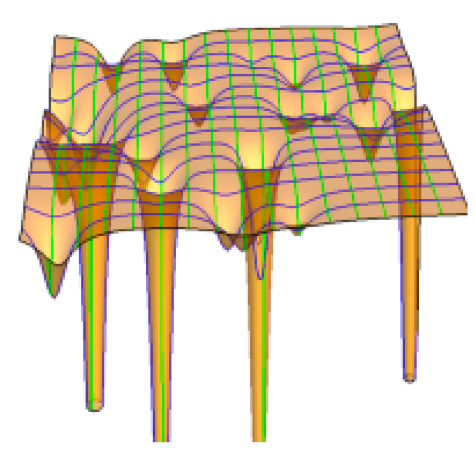
$$\mathcal{L}_\varepsilon := \{\sigma \in \mathcal{Q}_N | U(\sigma) \leq -\varepsilon N\}$$

Subset $C_\varepsilon \subset \mathcal{L}_\varepsilon$ called *edge-connected* $\Leftrightarrow$ pair $\sigma, \sigma' \in C_\varepsilon$ connected through an edge-path of adjacent edges.

Decompose $\mathcal{L}_\varepsilon = \bigcup_\alpha C_\varepsilon^\alpha$ into maximal edge-connected subsets C_ε^α.

For any $\varepsilon > 0$, the set $\mathcal{L}_\varepsilon$ does not percolate, i.e. there exists a subset $\Omega_{\varepsilon,N}$ of realizations such that:

1. $\mathbb{P}(\Omega_{\varepsilon,N}) \geq 1 - e^{-c_\varepsilon N}$ for some $c_\varepsilon > 0$
2. on $\Omega_{\varepsilon,N}$: $\max_\alpha |C_\varepsilon^\alpha| < K_\varepsilon = \lceil \frac{4 \ln 2}{\varepsilon^2} \rceil$



L. Faoro et al., arXive: 1812.06016

Decomposition of the Hamiltonian

$$H =: U_{\mathcal{L}_\varepsilon} \oplus H_{\mathcal{L}_\varepsilon^c} - \Gamma A_{\mathcal{L}_\varepsilon}$$

$U_{\mathcal{L}_\varepsilon}$ and $H_{\mathcal{L}_\varepsilon^c}$ restrictions of corresponding operators and $A_{\mathcal{L}_\varepsilon}$ is remainder term with matrix elements

$$\langle \sigma | A_{\mathcal{L}_\varepsilon} | \sigma' \rangle = \begin{cases} 1 & \text{if } \sigma \in \mathcal{L}_\varepsilon \text{ or } \sigma' \in \mathcal{L}_\varepsilon \text{ and } d(\sigma, \sigma') = 1, \\ 0 & \text{else.} \end{cases}$$

Upper bound for the operator norm: $\|A_{\mathcal{L}_\varepsilon}\| \leq \sqrt{2N \max_\alpha |C_\varepsilon^\alpha|}$

To conclude the upper bound, pick some $\varepsilon > 0$. The Golden-Thompson inequality yields

$$Z(\beta, \Gamma) \leq 2^{-N} e^{\beta\Gamma \|A_{\mathcal{L}_\varepsilon}\|} \left(\mathrm{Tr}_{\ell^2(\mathcal{L}_\varepsilon)} e^{-\beta U_{\mathcal{L}_\varepsilon}} + \mathrm{Tr}_{\ell^2(\mathcal{L}_\varepsilon^c)} e^{-\beta H_{\mathcal{L}_\varepsilon^c}} \right)$$

First term in the bracket: bounded by $Z(\beta, 0)$

Second term: all matrix elements of $-T$ are positive, this leads to the bound

$$\mathrm{Tr}_{\ell^2(\mathcal{L}_\varepsilon^c)} e^{-\beta H_{\mathcal{L}_\varepsilon^c}} \leq e^{\beta\varepsilon N} \mathrm{Tr} e^{-\beta\Gamma T}$$

On $\Omega_{\varepsilon,N}$ we thus get the following bound for all N large enough,

$$p_N(\beta, \Gamma) \leq \max\{p_N(\beta, 0), p^{\mathrm{PAR}}(\beta\Gamma)\} + 2\beta\varepsilon.$$

A Borel-Cantelli argument implies:

$$\limsup_{N\to\infty} p_N(\beta, \Gamma) \leq \max\{p^{\mathrm{REM}}(\beta), p^{\mathrm{PAR}}(\beta\Gamma)\} \text{ almost surely}$$

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