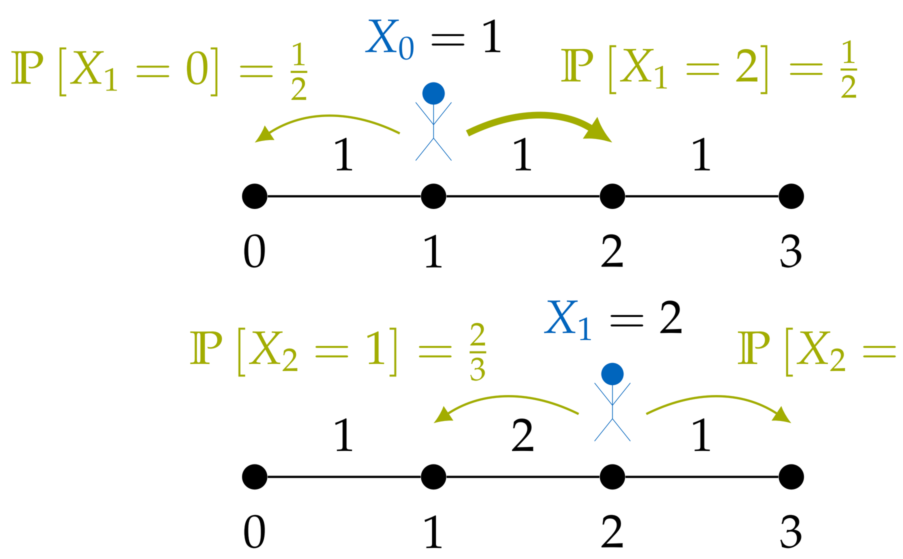
Variations on Reinforced Random Walks

Fabian Michel
Technical University of Munich

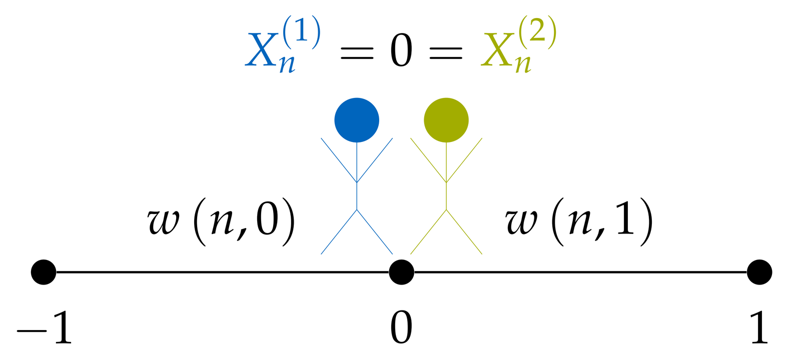
Reinforced Random Walks

We are interested in edge-reinforced random walks. In the basic setting, there is one random walker on a graph and the initial edge weights are 1. The walker then moves at discrete time steps and chooses the edge to traverse with probability proportional to the current edge weight. If an edge is traversed, its weight is increased by 1. This is called the linearly edge-reinforced random walk (LERRW).



2 Walkers on 3 Nodes

Consider the LERRW on a graph consisting of a 3-node segment. We now look at two walkers which both influence the edge weights. At every time step, the walker to move is chosen uniformly at random.



Edge weights at time n are denoted by $w(n,\cdot)$, positions of the walkers at time n by $X_n^{(\cdot)}$. Here, both walkers are in the central node 0, which is also their starting position. We look at the following quantities:

$$\tau_0 := 0 \qquad \tau_n := \inf\left\{k > \tau_{n-1} : X_k^{(1)} = X_k^{(2)} = 0\right\}$$

$$F_n := \frac{w(n,0)}{w(n,0) + w(n,1)} \qquad M_n := \frac{w(\tau_n,0)}{w(\tau_n,0) + w(\tau_n,1)} = F_{\tau_n}$$

Theorem 1 The random variables $M_n, n \geq 0$ form a martingale. F_n converges almost surely for $n \to \infty$, and the limit is identical to the limit of M_n.

Remark Note the similarity to the case with a single walker where the edge weights behave as the balls in a standard Pólya urn where two balls of the drawn color are added to the urn at every step.

Proof Idea In order to show the martingale property, look at

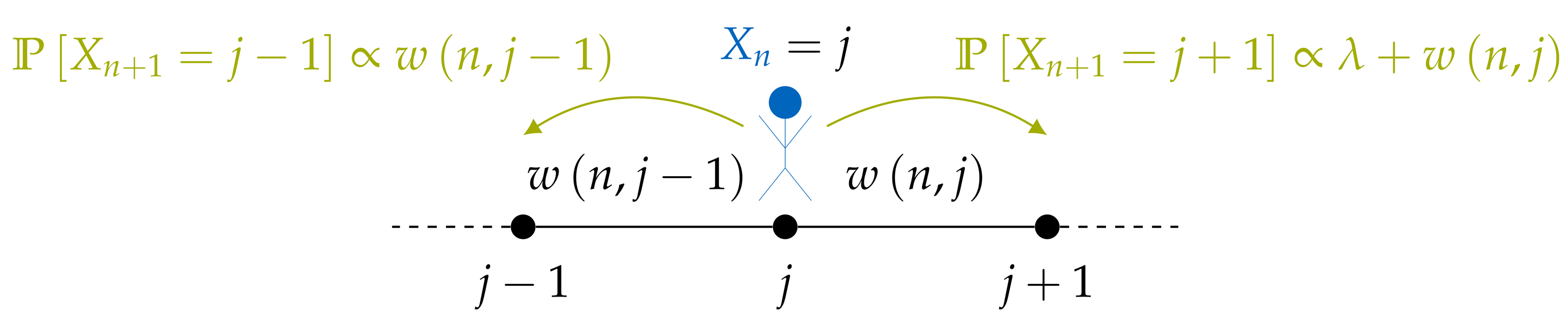
$$\mathbb{E}_{a,b,l} := \mathbb{E}\left[M_{n+1} \cdot \mathbb{1}_{\{\tau_{n+1} - \tau_n = 2l\}} \mid w(\tau_n, 0) = a, w(\tau_n, 1) = b\right]$$

Consider all possible paths of length $2l$ which the two walkers could take, and which end with both walkers meeting in the center at time $2l$ without having met in the center before. These paths can be extended to obtain all paths of length $2(l+1)$ with the same properties. This extension leads to a recursive formula allowing the calculation of $\mathbb{E}_{a,b,l}$. It then follows that the M_n form a martingale. The convergence of F_n follows by a Borel-Cantelli argument from the convergence of M_n. □

Conjecture 2 Define $Y := \lim_{n\to\infty} F_n = \lim_{n\to\infty} M_n$. Then $Y \in [0,1]$ has a density w.r.t. the Lebesgue measure on $[0,1]$. Y is not Beta-distributed.

Additive Bias

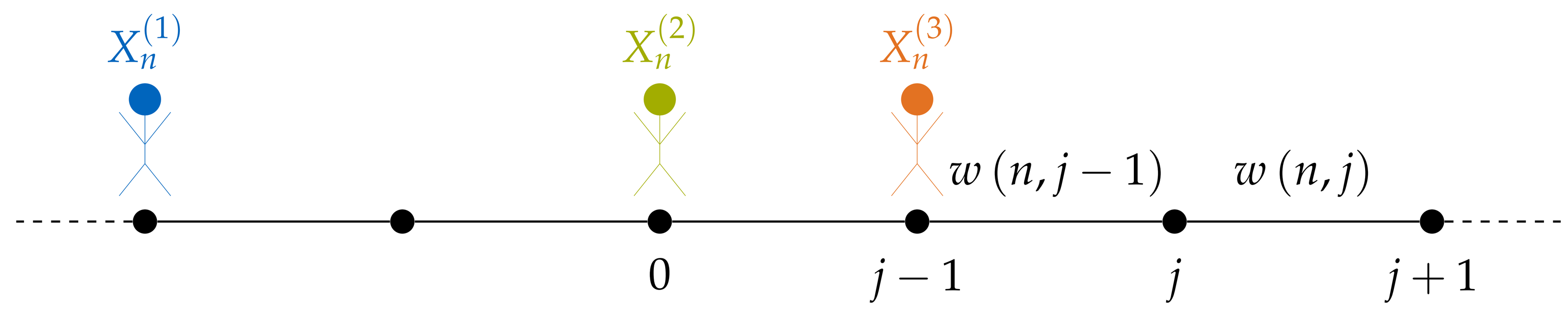
We can add a bias to the LERRW on $\mathbb{Z}$ by making the probability to go to the right proportional to λ plus the edge weight where $\lambda > 0$ is a parameter.



Theorem 3 The walk with additive bias λ is recurrent for $0 \leq \lambda \leq 1$ and transient for $\lambda > 1$. It has positive speed, if, and only if, $\lambda > 3$. If this is the case, then $\frac{X_n}{n} \to \frac{\lambda-3}{\lambda+1}$ almost surely. Otherwise, $\frac{X_n}{n} \to 0$ almost surely.

k Walkers on $\mathbb{Z}$

Consider k walkers on the graph $\mathbb{Z}$, where the walker to move is chosen uniformly at random at every step. We now (almost) only require that the weight of an edge is increased by an arbitrary amount (not necessarily 1) upon traversal, and that all but finitely many initial edge weights are 1.



We say that one of the walkers is recurrent if he visits every node of $\mathbb{Z}$ infinitely often, and we say that he has finite range if he only visits finitely many nodes.

Theorem 4 The edge-reinforced random walk with k walkers on $\mathbb{Z}$ satisfies:

$$\mathbb{P}\left[\forall m : X^{(m)} \text{ is recurrent}\right] + \mathbb{P}\left[\forall m : X^{(m)} \text{ has finite range}\right] = 1$$

Remark Again, note the similarity to the case with a single walker: depending on the reinforcement scheme, the walker is either recurrent or does have finite range. We did not prove it yet, but we conjecture that, given a certain reinforcement scheme, a single walker has finite range if, and only if, any finite number of walkers does have finite range with the same reinforcement scheme.

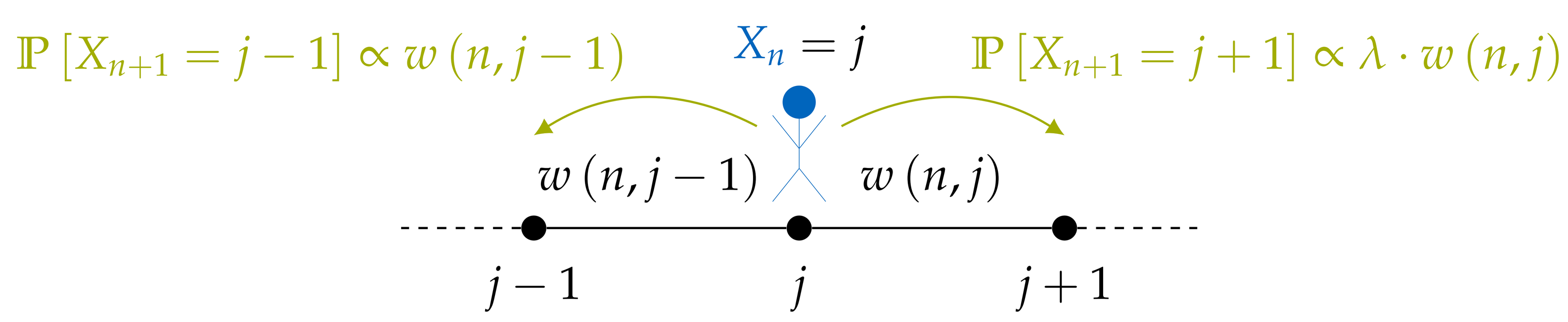
Proof Idea Let $\tau^{(m)} := \inf\left\{n \geq 0 : X_n^{(m)} \leq 0\right\}$. We will look at the quantity

$$M_n^{(m)} := \sum_{i=0}^{X^{(m)}_{n\wedge\tau^{(m)}}-1} \frac{1}{w(n \wedge \tau^{(m)}, i)}$$

$M_n^{(m)}$ sums up the inverse edge weights from 0 to the position of the m-th walker at time n, as long as the walker is to the right of 0. We can show that $M_n^{(m)}$ is a nonnegative supermartingale. We can add a nonnegative random variable to obtain a martingale which converges by the martingale convergence theorem.
Using that the value of the martingale changes by a fixed amount if a certain event occurs (which can therefore only happen finitely often by convergence), it is possible to show that every walker either returns to 0 or visits only finitely many nodes which have not been visited before by any other walker.
Using this, we can then show that either every node of $\mathbb{Z}$ is visited infinitely often by at least one of the walkers or all walkers have finite range. The final step consists in showing that any two walkers meeting infinitely often are either both recurrent or do both have finite range, which results in a proof of the theorem. □

Multiplicative Bias

We can add a bias to the LERRW on $\mathbb{Z}$ by making the probability to go to the right proportional to λ times the edge weight where $\lambda > 0$ is a parameter.



Theorem 5 Consider the walk with multiplicative bias $\lambda \in \mathbb{Q}_{>0}$. If $j \in \mathbb{Z}$ is visited infinitely often and $\lambda > 1$, then $\frac{\lambda w(n,j)}{w(n,j-1)+\lambda w(n,j)} \to 1$ almost surely.

Lemma 6 If the walk with multiplicative bias λ visits at least one node infinitely often, then all nodes are visited infinitely often almost surely.

Conjecture 7 The walk with multiplicative bias λ is transient for $\lambda \neq 1$.

Selected References

[1] Robin Pemantle. *A Survey of Random Processes with Reinforcement*. Probability Surveys, Vol. 4, pp. 1-79. Institute of Mathematical Statistics and Bernoulli Society, 2007. Accessible at https://arxiv.org/abs/math/0610076.

[2] Burgess Davis. *Reinforced Random Walk*. Probability Theory and Related Fields, Vol. 84, pp. 203-229. Springer-Verlag, 1990. Accessible at https://link.springer.com/article/10.1007/BF01197845.