TopMath
Mathematik mit Promotion

Approximation of Conformal Mappings on the GPU

Tim Reinhardt

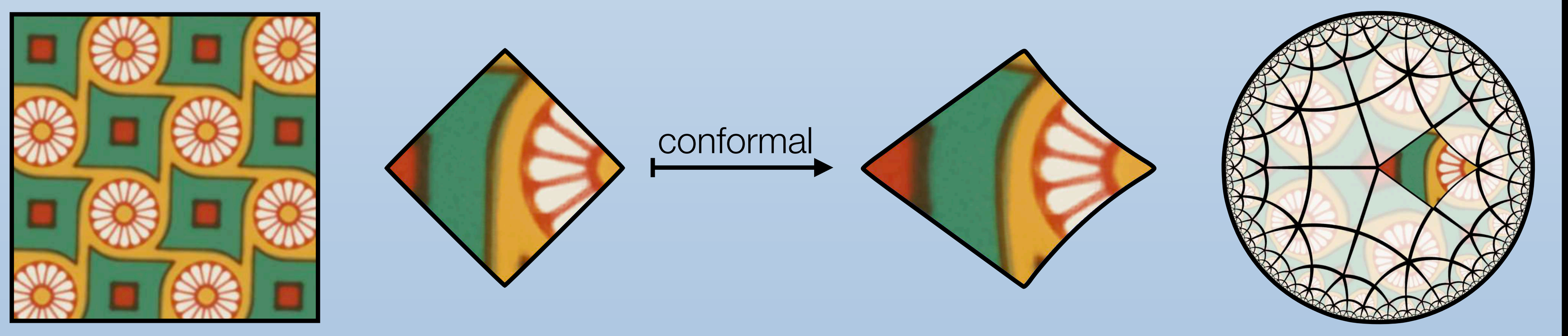
Abstract

We discuss an algorithm for the computation of conformal mappings between simply connected regions in the complex plane $\mathbb{C}$ which are bounded by Möbius arcs. Specifically, we introduce an iteration scheme that frequently approximates conformal maps well. Two guiding themes play a critical role: parallelizability and the Schwarz reflection principle. The former allows us to utilize the graphic card for implementations and thereby to visualize the algorithm's convergence behavior in real time. The latter is a method how to treat the boundaries of the regions.
Building on classical statements about conformal mappings from complex analysis, we discuss possibilities to concretely state conformal maps and to discretize the underlying theory. This allows us to apply results from discrete differential geometry to the algorithm and to experimentally compare it with known mappings.

Conformal Maps

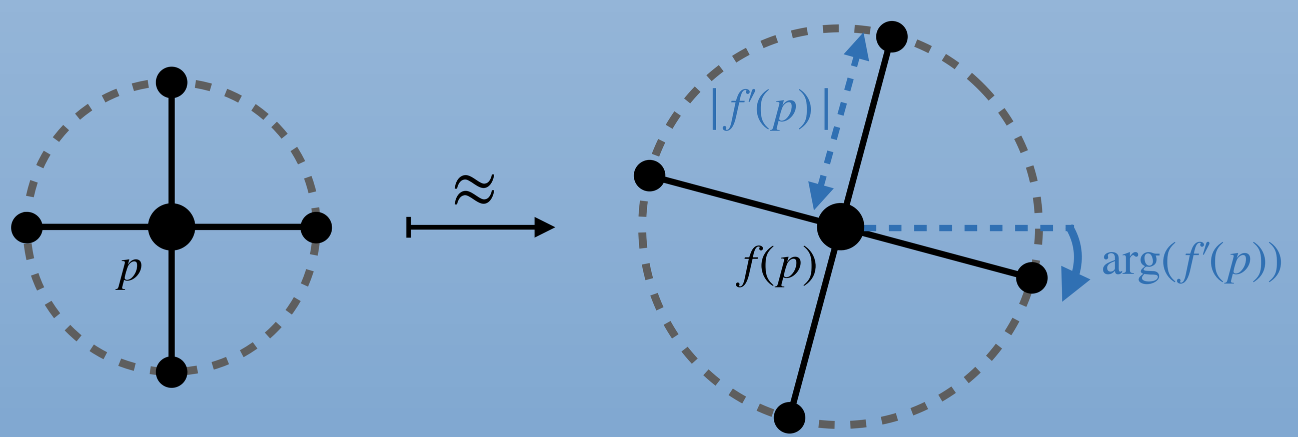
A function $f\colon \mathbb{C} \to \mathbb{C}$ is called *conformal* if it is angle- and orientation-preserving $\Leftrightarrow$ f holomorphic and $f' \neq 0$.

One application lies in constructing tilings of the hyperbolic plane. These can be obtained by conformal deformation of fundamental cells of Euclidean tilings. This approach preserves the artistic content of the cell as conformal maps locally act like scale-rotations. [1]



GPU-Approximation of Conformal Maps

Necessary Condition for Conformality. Conformal maps are infinitesimal scale-rotations, so each image point $f(p)$ under a conformal map f is approximately the arithmetic mean of the images of close neighbours of the preimage point p.



Basic Idea for Algorithm.
- Initialize a pixel grid with a reasonable mapping.
- Iteratively force the mapping to fulfil the above necessary condition by defining $f(p)$ as the arithmetic mean of the images of the points in the van Neumann neighborhood of p.
- Perform the iteration scheme parallelized on the GPU.

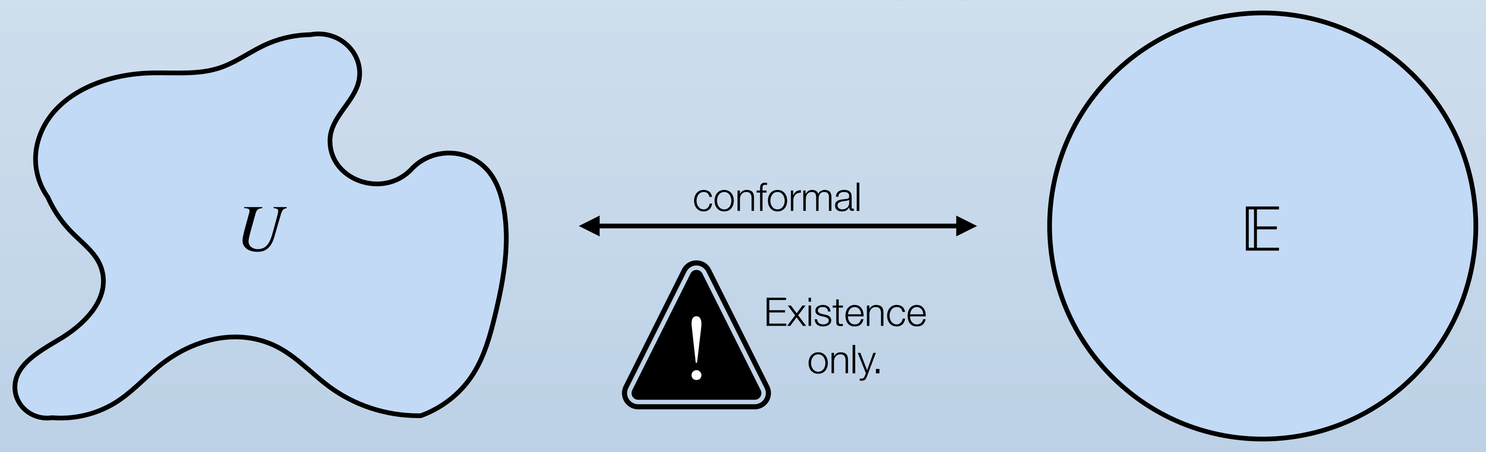
How to Deal with Boundaries?
There are situations close to the boundaries of the considered regions in which neighborhood points of some pixels p are not contained in the region. In this case, we can analytically continue f from the interior to the exterior of the region by straight-line or circular reflection across both the corresponding boundary segments of the image and preimage domain. This is the so-called *Schwarz reflection principle*. [2]

Selected References

[1] J. Richter-Gebert and M. von Gagern. “Hyperbolization of Euclidean Ornaments”. In: *The Electronic Journal of Combinatorics* 16.2 (2009).
[2] Z. Nehari. *Conformal Mapping*. Dover Publications, Inc., 1975.
[3] A. Bobenko, U. Pinkall, and B. Springborn. “Discrete conformal maps and ideal hyperbolic polyhedra”. In: *Geometry & Topology* 19.4 (2015).

Riemann Mapping Theorem

Every simply connected region $U \subsetneq \mathbb{C}$ is biholomorphically equivalent to the unit disk $\mathbb{E} = \{z \in \mathbb{C} \colon |z| < 1\}$. [2]



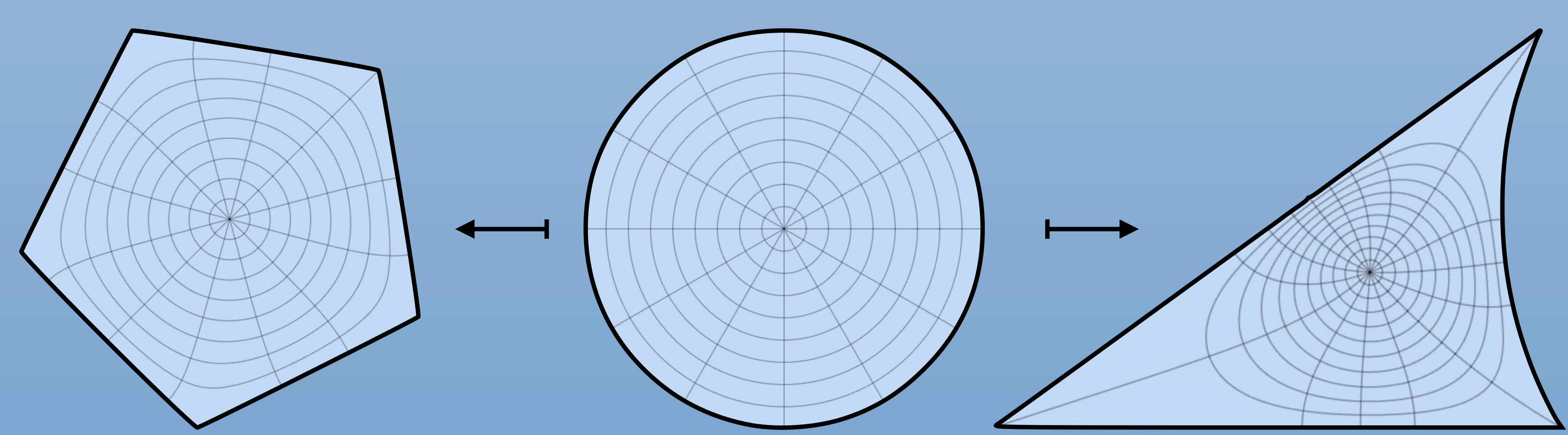
Explicit Forms of Conformal Maps

Schwarz-Christoffel-Mappings. Let $P \subset \mathbb{C}$ be a straight-line polygon. The map $f\colon \mathbb{E} \to P$,

$$f(z) = \int_0^z (1 - \zeta^n)^{-2/n}\, \mathrm{d}\zeta$$

conformally maps the unit disk to the polygon's interior. [2]

Hypergeometric Functions. These power series can be used to construct conformal maps from the unit disk to triangles which are bounded by straight-line or circular segments. [2]



Results

Idea. Compare algorithm-results with known exact conformal maps.
Problem. GPU comes with low calculation precision (8 bits).
Solution. Perform numerical experiments with a non-parallelized (i.e. slow) but precise implementation of the algorithm on the CPU.

Results. The fast, low-precision GPU-approximation returns visually satisfying results while the slow, high-precision CPU-algorithm approximates the exact maps much better in the examined instances and potentially yields arbitrary precision for increasing iterations.
Moreover, invoking the concept of *discrete conformal equivalence*, which can be characterized in terms of length-cross-ratios on a triangulation, a local error estimate for the approximating maps can be formulated. This may be a suitable way to evaluate outputs of the algorithm without knowing the exact map. [3]