Upper and lower bounds on the Flatness Constant

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Lattices in $\mathbb{R}^n$

A set $\Lambda \subset \mathbb{R}^n$ is called a *Lattice*, if it has the following properties.

- Λ is a group with respect to vector addition, meaning that:
  - Λ contains the origin
  - If $x, y \in \Lambda$, then $x + y \in \Lambda$
  - If $x \in \Lambda$, then $-x \in \Lambda$
- Λ is discrete, meaning that for each $x \in \Lambda$, there is an $\varepsilon > 0$ such that the ball with radius ε and center x only intersects Λ in x
- Λ is fulldimensional, meaning that the linear hull of Λ is $\mathbb{R}^n$

A useful property of lattices is that every n-dimensional lattice can be mapped onto the integer lattice $\mathbb{Z}^n$ using a bijective linear map. This allows us to generalize many results that we obtain on the integer lattice to general lattices.

Lattice Width

The *dual lattice* Λ^* of a lattice $\Lambda \subset \mathbb{R}^n$ is defined as the set $\{x \in \mathbb{R}^n : x^T y \in \mathbb{Z} \text{ for all } y \in \Lambda\}$. Particularly, $\mathbb{Z}^n$ is its own dual lattice. The *lattice width* of a convex body $K \subset \mathbb{R}^n$ with respect to the lattice Λ is then calculated as

$$\mathrm{lw}(K) := \min_{v \in \Lambda^* \setminus \{0\}} \{\max_{x \in K} v^T x - \min_{x \in K} v^T x\}.$$

One way to visually grasp this concept is to measure the minimum number of parallel layers in the lattice that are sufficient to cover K.

An Example for Lattice Width

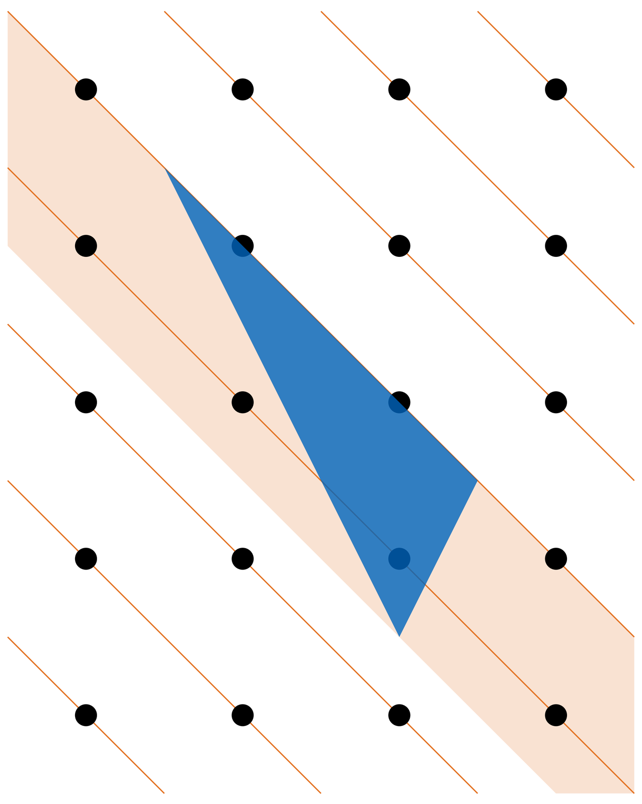

Figure 1: A triangle embedded in $\mathbb{R}^2$ that has lattice width 1.5 with respect to $\mathbb{Z}^2$. Note that if one chooses any other direction for parallel lattice lines to dissect $\mathbb{R}^2$ into layers, the triangle will cover more than 1.5 of these layers.

Lattice Freeness

A convex body $K \subset \mathbb{R}^n$ is called *lattice-free* with respect to the lattice Λ, if the interior of K does not contain any lattice points of Λ. In particular, if the feasible region of a system of linear inequalities is lattice-free with respect to $\mathbb{Z}^n$ and one tightens the bounds of all linear inequalities by an arbitrarily small amount, it follows that the system has no integer solutions. Due to this relation, the concept of lattice-freeness has numerous applications in the field of discrete optimization.

History of the Flatness Constant

In 1948, Khinchine proved the following statement known as the *Flatness Theorem*:

> For any dimension n, there is a fixed value $\mathrm{Flt}(n)$ such that any lattice-free convex body K has lattice width less or equal to $\mathrm{Flt}(n)$.

The value $\mathrm{Flt}(n)$ is called the *Flatness Constant* in dimension n. Note that we didn't specify the lattice, since this statement holds (and the Flatness Constant is the same) for any lattice $\Lambda \subset \mathbb{R}^n$.

The precise value of the Flatness Constant is still unknown. Important milestones for improving the upper bound include:

- $\mathrm{Flt}(n) \leq (n+1)!$, the initial bound, found by Khinchine
- $\mathrm{Flt}(n) \leq n^{5/2}$, the first polynomial bound, found by Lagarias, Lenstra and Schnorr
- $\mathrm{Flt}(n) \leq O(n \log^3(2n))$, the currently best known bound, found by Reis and Rothvoß

Flatness constant in dimension 2

We say that a lattice-free body is a *maximal lattice-free body*, if it is not strictly contained in any lattice-free body. A result by Lovász states that any maximal lattice-free body K has the following properties:

- K is a polytope with 2^n or less facets
- Each facet of K contains at least one lattice point in its relative interior

In the 2-dimensional case, this allows us to restrict ourselves to triangles and quadrilaterals. In 1989, Hurkens proved that lattice-free quadrilaterals have lattice width at most 2, and lattice-free triangles have lattice width less or equal to $1 + \frac{2}{\sqrt{3}}$. By giving an explicit construction for a lattice-free triangle which achieves this lattice width, it is proven that $\mathrm{Flt}(2) = 1 + \frac{2}{\sqrt{3}}$.

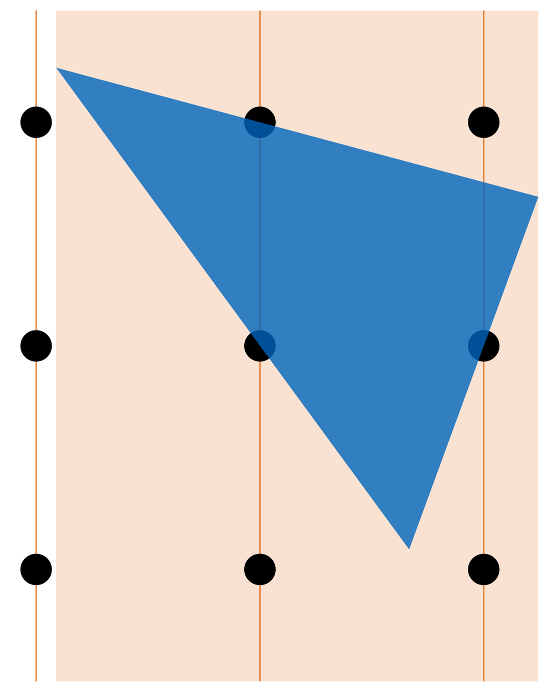

Figure 2: The (up to symmetry) unique maximizer for lattice width in dimension 2, and a visualization of a possible direction along which the triangle attains its lattice width.

Lower Bounds on the Flatness Constant

The trivial lower bound to the Flatness Constant is $\mathrm{Flt}(n) \geq n$. To see this, note that scaling the n-dimensional unit simplex by the factor n results in a lattice-free body with lattice width n.

In general, any body $K \in \mathbb{R}^n$ with lattice width Δ can be extended to a body $K^* \in \mathbb{R}^{nm}$ with lattice width $m \cdot \Delta$ by first translating K to achieve $0 \in K$ and then setting

$$K^* = \left\{ (\lambda_1 x_1, \ldots, \lambda_m x_m) \subset \mathbb{R}^{nm} \mid \sum_{i=1}^{m} \lambda_i = 1; \lambda_1, \ldots, \lambda_m \geq 0; x_1, \ldots, x_m \in m \cdot K \right\}.$$

Therefore, any lattice-free body $K \subset \mathbb{R}^n$ that gives a new highest known value to $\mathrm{lw}(K)/n$ provides a new asymptotical lower bound for the Flatness Constant in high dimensions.

Codenotti and Santos first mentioned a 3-dimensional body with lattice width $2 + \sqrt{2}$, which, using this technique, provides the asymptotical lower bound $\mathrm{Flt}(n) \geq \frac{2+\sqrt{2}}{3} \cdot n \approx 1.138 \cdot n$.

A construction by Mayrhofer, S. and Weltge yields a sequence of n-dimensional lattice-free simplices Δ_n for $n \in \mathbb{N}$ that achieve $\mathrm{lw}(\Delta_n) \to 2n$ for $n \to \infty$.

References

[1] G. Codenotti and F. Santos. Hollow polytopes of large width. *Proceedings of the American Mathematical Society*, 148, 2020.

[2] C.A.J. Hurkens. Blowing up convex sets in the plane. *Linear Algebra and its Applications*, 134, 1990.

[3] A. Khinchine. A quantitative formulation of kroneckers theory of approximation. *Izd. Akad. Nauk SSSR*, 12, 1948.

[4] J.C. Lagarias, H.W. Lenstra, and C.P. Schnorr. Korkin-zolotarev bases and successive minima of a lattice and its reciprocal lattice. *Combinatorica*, 10, 1990.

[5] L. Lovász. Geometry of numbers and integer programming. *Mathematical programming: Recent developments and applications*, 1989.

[6] L. Mayrhofer, J. Schade, and S. Weltge. Lattice-free simplices with lattice width 2 d-o (d). In *International Conference on Integer Programming and Combinatorial Optimization*, 2022.

[7] V. Reis and T. Rothvoss. The subspace flatness conjecture and faster integer programming. *arXiv preprint arXiv:2303.14605*, 2023.