Geometric Inequalities Involving Different Diameter Definitions

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Notation

$\mathcal{C}^n := \{C \subset \mathbb{R}^n,\ C$ compact convex$\}$, $\mathcal{C}_0^n := \{C \subset \mathcal{C}^n,\ 0 \in \operatorname{int} C\}$;
$K \in \mathcal{C}^n, C \in \mathcal{C}_0^n$ always.
Polar: $C^\circ = \{a \in \mathbb{R}^n : a^T x \le 1,\ x \in C\}$
Support function: $h_C(\cdot) : \mathbb{R}^n \to \mathbb{R},\ h_C(a) := \max_{x \in C} a^T s$
Gauge function: $\|x\|_C = \min\{\lambda \ge 0 : x \in \lambda C\}$
Circumradius: $R(K, C) := \min\{\rho \ge 0 : \exists t \in \mathbb{R}^n$ such that $K \subset t + \rho C\}$
Inradius: $r(K, C) := \max\{\rho \ge 0 : \exists t \in \mathbb{R}^n$ such that $t + \rho C \subset K\}$

Diameter Definitions

- *Minimum diameter*:
$$D_{\mathrm{MIN}}(K, C) = \max_{x,y \in K} \|x - y\|_C$$
- *Harmonic diameter*:
$$D_{\mathrm{HM}}(K, C) = \max_{x,y \in K} \frac{1}{2} (\|x - y\|_C + \|y - x\|_C)$$
- *Arithmetic diameter* (standard diameter):
$$D_{\mathrm{AM}}(K, C) = \max_{x,y \in K} 2R([x, y], C) = \max_{s \in \mathbb{R}^n \setminus \{0\}} 2 \cdot \frac{h_K(s) + h_K(-s)}{h_C(s) + h_C(-s)}$$
- *Maximum diameter*:
$$D_{\mathrm{MAX}} = \max_{s \in \mathbb{R}^n \setminus \{0\}} \frac{h_K(s) + h_K(-s)}{\max\{h_C(s), h_C(-s)\}}$$
- If C 0-symmetric
$$D(K, C) = D_{\mathrm{MIN}}(K, C) = D_{\mathrm{HM}}(K, C) = D_{\mathrm{AM}}(K, C) = D_{\mathrm{MAX}}(K, C)$$

Symmetrizations

Symmetrizing the second argument:
- $D_{\mathrm{MIN}}(K, C) = D(K, C \cap (-C))$
- $D_{\mathrm{HM}}(K, C) = D(K, (\frac{C^\circ - C^\circ}{2})^\circ)$
- $D_{\mathrm{AM}}(K, C) = D(K, \frac{C - C}{2})$
- $D_{\mathrm{MAX}} = D(K, \operatorname{conv}(C \cup (-C)))$

Symmetrizations:
- *minimum* $C_{\mathrm{MIN}} := C \cap -C$
- *harmonic mean* $C_{\mathrm{HM}} := \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$
- *arithmetic mean* $C_{\mathrm{AM}} := \frac{C - C}{2}$
- *maximum* $C_{\mathrm{MAX}} := \operatorname{conv}(C \cup -C)$

Notation: $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}, \mathrm{AM}, \mathrm{MAX}\}$

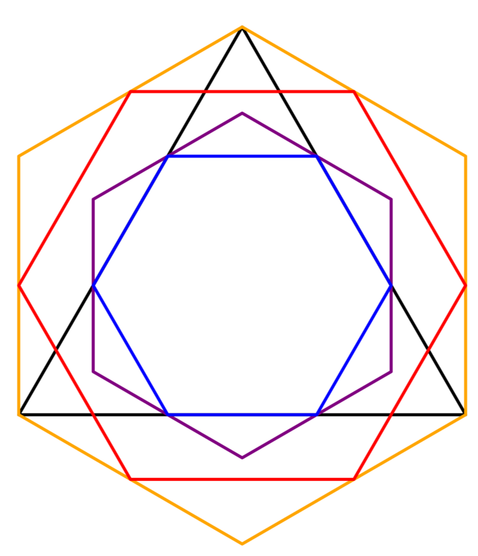


Fig. 1: The regular triangle (black) and its symmetrizations: minimum (blue), harmonic mean (purple), arithmetic mean (red), maximum (orange).

Extended harmonic-arithmetic mean inequality:
$$C_{\mathrm{MIN}} \subset C_{\mathrm{HM}} \subset C_{\mathrm{AM}} \subset C_{\mathrm{MAX}},$$
with equality between any of the means if and only if $C = -C$.
Centering the gauge:
Minkowski-asymmetry: $s(C) := R(C, -C)$
C is *Minkowski-centered* if $C \subset -s(C)C$.

Completion of the Gauge

- K *complete*: every $\tilde{K} \supset K$ has larger diameter
- *Completion*: $K^* \supset K$, complete, with $D_{\mathrm{M}}(K, C) = D_{\mathrm{M}}(K^*, C)$
- Gauge C always complete for $\mathrm{M} = \mathrm{AM}$ or if $C = -C$.
- C_{M} complete w. r. t. C using D_{M}.
- Let $C \in \mathcal{C}_0^n$ be Minkowski-centered. The following are equivalent:
  i) $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ is a completion of C using D_{HM},
  ii) $R(C_{\mathrm{AM}}, C_{\mathrm{HM}}) = \frac{s(C)+1}{2}$ and
  iii) $D(C, C_{\mathrm{HM}}) = 2R(C, C_{\mathrm{HM}})$.
- C_{MAX} is always a completion of C using D_{MAX}.
- ρC_{MIN} is a completion of C using D_{MIN} iff $\rho = s(C) = 1$ $(C = -C)$.

Blaschke-Santaló Diagrams

Given a gauge $C \in \mathcal{C}_0^n$ and values (r, R, D), is there a convex body $K \in \mathcal{C}^n$ such that its inradius w. r. t. C is r, its circumradius is R, and its diameter is D?
Blaschke-Santaló diagram: $f_{\mathrm{M}}(\mathcal{C}^n, C)$ with
$$f_{\mathrm{M}} : \mathcal{C}^n \times \mathcal{C}_0^n \to \mathbb{R}^2,\ f_{\mathrm{M}}(K, C) = \left(\frac{r(K, C)}{R(K, C)}, \frac{D_{\mathrm{M}}(K, C)}{2R(K, C)}\right)$$

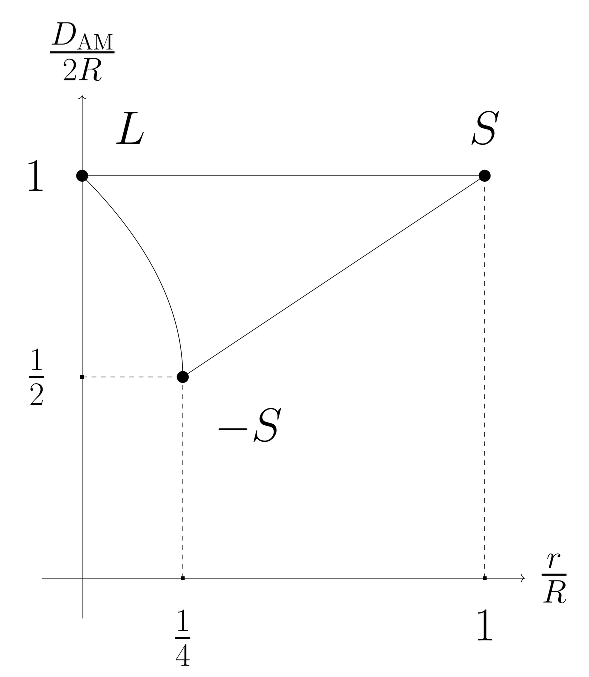
Diagrams for triangular gauges
- $f_{\mathrm{AM}}(\mathcal{C}^2, S) = f_{\mathrm{AM}}(\mathcal{C}^2, \mathcal{C}_0^2)$ given for triangular gauges S in [1].
- $f_{\mathrm{M}}(\mathcal{C}^2, S)$ for the remaining diameters and Minkowski-centered triangular gauges S.

Inequalities

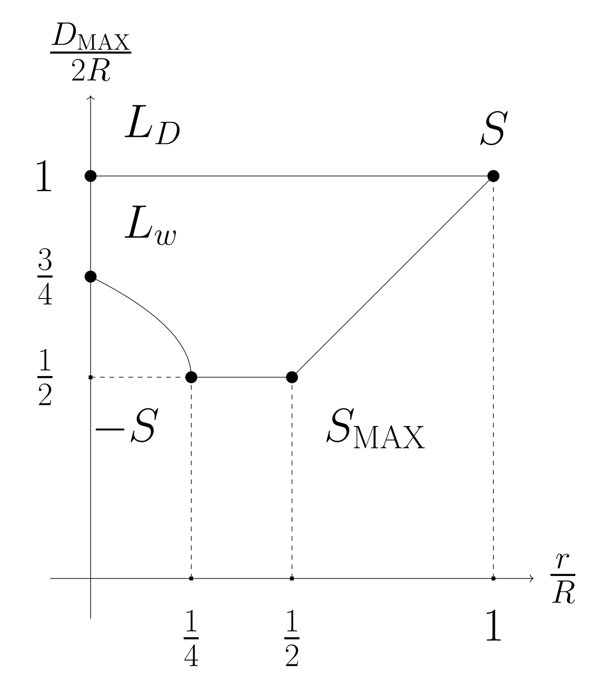
For example with $\mathrm{M} = \mathrm{MAX}$:

Theorem 1 *For every Minkowski-centered triangle S the diagram $f_{\mathrm{MAX}}(\mathcal{C}^2, S)$ is fully described by the inequalities*
$$\frac{D_{\mathrm{MAX}}(K, S)}{2} \le R(K, S)$$
$$r(K, S) \le \frac{D_{\mathrm{MAX}}(K, S)}{2}$$
$$0 \le r(K, S)$$
$$R(K, S) \le D_{\mathrm{MAX}}(K, S)$$
$$\left(\frac{D_{\mathrm{MAX}}(K, S)}{R(K, S)} - \frac{1}{2}\right)\left(\frac{3}{2} - \frac{D_{\mathrm{MAX}}(K, S)}{R(K, S)}\right) \le \frac{r(K, S)}{R(K, S)}.$$

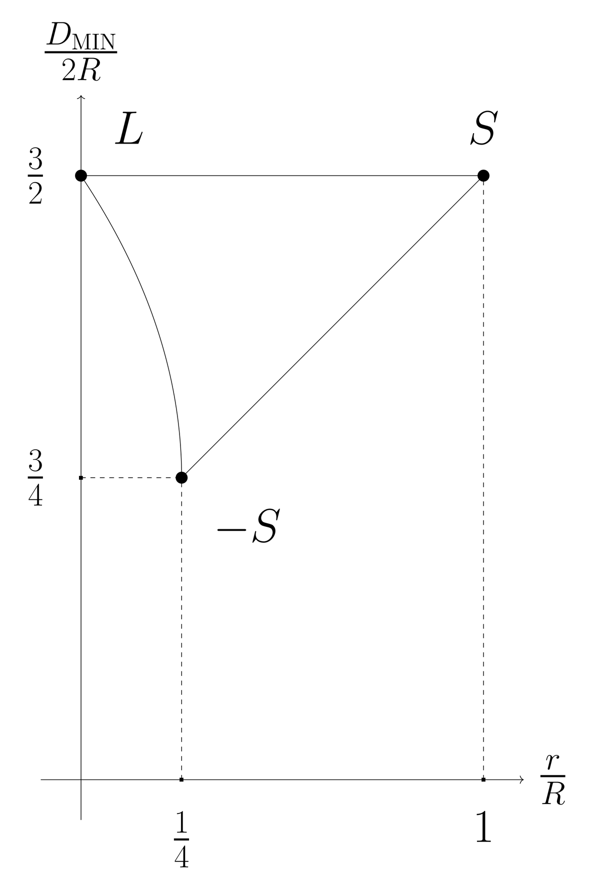
$f_{\mathrm{AM}}(\mathcal{C}^2, S)$



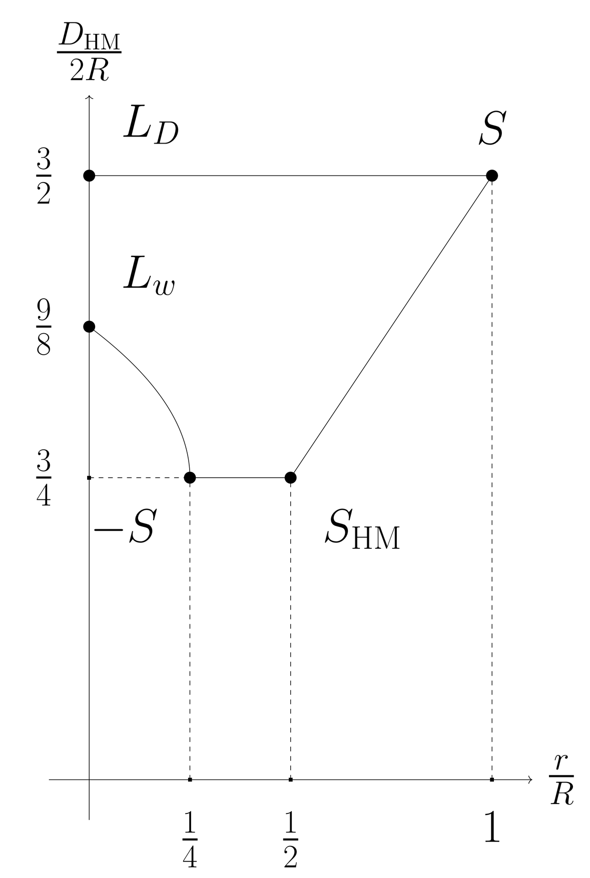
$f_{\mathrm{MAX}}(\mathcal{C}^2, S)$



$f_{\mathrm{MIN}}(\mathcal{C}^2, S)$



$f_{\mathrm{HM}}(\mathcal{C}^2, S)$



References

[1] R. Brandenberg and B. González Merino. "Behaviour of inradius, circumradius, and diameter in generalized Minkowski spaces". In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 116.3 (2022), p. 105.

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